The artful kaleidoscopes of the circular and spherical bells.

I am happy to finally come to BRIDGES to share this presentation that summarizes my improbable research. I hope you will enjoy it.

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It all started learning a simple game of chance introduced by Barnsley as explained here. To begin, select a triangle numbering the vertices according to the sides of a die. Then, mark the point in the middle of the line joining 1--2 and 3--4 and roll a die.

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Suppose that the outcome is 5. Then, mark the point that lies in the middle of the initial little square and the vertex in the triangle marked with the 5. Now, keep repeating, moving from where you are to the middle of the vertices given by the outcomes of the die.

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After several iterations, say 500, this is what appears.

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After 8,000, it becomes clear that the procedure converges to a repetitive pattern: the famous Sierpinski gasket, a whole triangle with its middle triangles excluded, ad infinitum.

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As iterations produce interesting objects, we may change the rules and employ the two simple maps shown here, that also take a point in the plane into another point in the plane, as before.

As these are the key equations needed to follow the lecture, please allow me to describe them in some detail. First, notice that the x components are decoupled from the y components. While rule w sub 1 just divides the input value by 2, rule w sub 2 divides by 2 and then adds to it one half.

Second, observe that the y components, for both maps, are linear combinations of the input x and y values. While a parameter d sub 1 multiplies the y value on the first map, a parameter d sub 2 does so on the second map.

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Now more details. While w sub 1 (in blue) takes the point (0,0) into (0,0) and w sub 2 (in red) (1,0) into (1,0), w sub 1 takes the point (1,0) into (1/2,1) and w sub 2 (0,0) also into (1/2,1), as shown.

As seen, w sub 1 operates to the left and w sub 2 to the right of the domain, which goes from 0 to 1, and the question becomes: what is it that the iteration of these rules generates?

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We may now play the chaos game with these simple rules, having, as an example, d sub 1 equal to minus d sub 2 equal to z equal to 0.5, and using a coin to decide where to move, say w sub 1, to the left, if heads and w sub 2, to the right, if tails.

This page illustrates the beginning of the game using the middle point marked bold.

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Imagine that heads happen first. Then we use map w sub 1 to find a new point to the left, having coordinates one quarter and one half plus one half, or a quarter one, as marked.

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Toss the coin again, and say heads happen again. Then, move from where you were into the other point to the left, as shown.

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Toss the coin another time and imagine it is tails. Then move from where you were into the point shown on the right using w sub 2. And so on...

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Here it is what is found after 100 iterations.

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And here is the final attractor after 10,000 tosses. As if by magic, the bombardment of dots produced arranges into a single object shapes as a wire, and this set, also shaped as a mountain profile, turns out to be found irrespective of chance and of the type of coin used, either fair or biased.

Page 14.

It happens that, as found by Barnsley, the ideas give rise to other interesting profiles.

This page shows alternative shapes as defined by sign combinations of the parameters d sub 1 and d sub 2, while maintaining their magnitude z equal to 0.5 on the left and 0.8 on the right, and for constructions that preserve the initial points mentioned before (0,0), (0.5,1) and (1,0).

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While cases + -, that is d sub 1 positive and d sub 2 negative, - + yielding just the same pattern flipped and - - corresponds to mountains, case + +, when both d sub 1 and d sub 2 are positive, give clouds.

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As shown comparing the sets for distinct magnitudes, the profiles on the right require more ink, as they are clearly thicker. Such convoluted wires fill increasingly more space as z increases beyond 0.5, and they have the property that their lengths become infinite. When z tends to 1, such wires fill two-dimensional space.

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As mentioned, the same pattern is obtained for any type of coin, fair or biased, but the chaos game induces distinct stable histograms over the set depending on how it is filled: a uniform one when using a fair coin and one containing many thorns for the biased case, which corresponds to going 70% to the left and 30% to the right.

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One good day it became relevant to me to study how such histograms look over the x and y directions. Let's see first over x.

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What is seen over x is shown here by the middle of this page: the collection of thorns portrayed along a line. The structure of such a set is easily understood by following the dynamics of the chaos game.

Before the game starts all values are equally likely, but, once the biased coin is tossed, such changes the landscape as now there is 70% chance of being to the left yielding a higher histogram to that side. As the game progresses, the sub-domains split in two according to the same proportions, thus defining a cascade that ultimately generates the collection of thorns.

Page 20.

As seen, such thorns define layers and they intertwine in such a way that thorns corresponding to the same level do not touch. As such, the layers are supported by sets so fractured that they have the structure of dust.

Page 21.

It happens that such an object is also related to the way turbulence happens in nature.

When the inertia of a fluid subjugates the fluid's cohesion, that is when its Reynolds number shown here is sufficiently high, the fluid breaks into an irreversible chain of inwardly rotating eddies, which divide into eddies, that divide into eddies, and so on until dissipation. Such a process, not always having the most massive eddies to the left but alternating left and right, leads to an uneven and intermittent concentration of energies arranged into layers, given precisely and surprisingly by the 70-30 proportion.

Page 22.

The spiky object, being the product of multiple fragmentations, is known as a mulifractal and the layering in them is also useful to study the biased wealth inequalities of nations.

Page 23.

Let's now study how the collection of thorns looks over y. This is more complicated as from a single point of view we perceive at least two thorns that need to be added.

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In the spirit of the discussion, here is a mountain wire and the corresponding histogram over x, based on a 70-30 coin, and called dx.

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Now here is the obtained histogram over y, called dy.

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The wire may be interpreted as a system in which dx is the input and dy is the output.

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In a Platonic sense, dy is the “shadow” we see when illuminating the wire by dx.

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Given what dx represents, the object dy may be interpreted as a transformation of turbulence.

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Interestingly, dx appears random, but it is not as the building blocks are fully determined. These Platonic-romantic ideas, combining wires and multifractals, turn out to be useful to model a host of geophysical patterns. This is the reason they call me professor at UC Davis.

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As the essence of this talk lies in a limit, here it is what is found in the + - mountain case for a 70-30 illumination when z tends to one.

The wire grows such that it densely fills two-dimensional space and, in the process, it transmutes the thorns over dust in the input multifractal into a harmonious and smooth output shaped as a bell.

Page 31.

This surprising result happens to be universal as Gaussian bells with finite centers are found from the same wire, not only while using fair or arbitrarily biased coins, but also for any non-discrete input dx.

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As the bell is associated with the conduction of heat, the results define an unforeseen bridge from dissipation to conduction or from turbulent violence to diffusive calmness—and not the other way around—, as it is usually studied like from order to chaos.

Page 33.

Clearly, infinity is behind this magical limiting case and it happens that the system has the peculiar property that any small portion of it also defines bells.

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The ideas may be extended to higher dimensions, iterating simple maps having more coordinates.

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Such give rise to a new set of parameters in polar coordinates having radial and angular domains.

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And the iteration of such maps yields wires from one to two or from one to three dimensions.

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As illustrated here on the left, non-trivial wires in 3D yield interesting shadows over two dimensions, whose shapes add to the Platonic notions to model complexity.

As is seen on the right, the ideas may yield, from a wire in four dimensions, complex patterns over three dimensions that may represent pollution and other geophysical data in space.

Page 38.

It occurs that Gaussian shapes are also found for wires defined over higher dimensions.

In a suitable limit, when the norm of the parameter matrices tend to one, the corresponding wires fill up now volumes and, in this superior dimension, they cast shadows of bells independently of non-discrete illuminations dx.

As is illustrated here, a multifractal input dx --drawn in the lower center-- enlightens a limiting wire from x to the plane (y, z), depicted to the right in its two components from x to y and from x to z, to define a two-dimensional circular bell on the left, as shown from above in dyz and on the sides in dy and dz.

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Even though circular bells are the most common, depending on the signs of radial parameters the limiting projection may also be either an elliptical bell or an oscillation between many bells, whose centers dance around or inside a circle.

These Platonic notions generate in the limit Gaussians everywhere, a notation inspired by Barnsley's lovely book “Fractals everywhere.”

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Interestingly, even if one were to consider only half of such a wire, one would still encounter bells and this regenerating property, as previously described for the + - mountain, happens to be true for even smaller parts, ad infinitum.

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Eventually, we proved the Gaussian result in the one-dimensional case, but the two-dimensional counterpart remains elusive, even to this day.

As such, we decided to study how the concentric circles were formed, drawing not the final summary of all the iterations as done here, but rather plotting successive groups of, say, 10,000 points.

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To illustrate the surprise in store, here is a sequence of successive patterns when all angular parameters equal 60 degrees and for a choice of radial parameters close to one, while doing the iterations following the binary expansion of pi.

Yes, this is an artful kaleidoscope of patterns making the circular bell, where the colors denote the map used, as in the beginning of the presentation.

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The iteration of simple linear maps defines treasures inside the bell.

Yes, the superposition of the patterns on the left--for angular parameters equal to 90 degrees--, and many more not shown, gives a circular bell.

And the same happens on the right for angles of 60 degrees, which include diatoms and gems.

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These two galleries of rosettes are just examples of infinitely many mandalas that magically interlock with one another to form perfect circles and truthful bells.

The geometries obtained depend on the precise sequence used to guide the iterations, that is, on the specific outcomes of a coin (or of pi), and on other parameters that dictate the number of tips the patterns have.

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As is seen, in this rather geometric central limit, and only in the limit, there is hidden order in chance. Next time you hear a bell just remember that such reflects incredible beauty!

Page 46.

The treasures are certainly varied, and we know by now that all ice crystals, as on the left, and several biochemical rosettes, including the one of DNA, as on the right, are mathematical designs living inside the bell.

Page 47.

These ice crystals were grown filling up templates of photographed flakes using limiting maps yielding patterns with six tips, as in nature via the process of diffusion.

Their growth and variety evoke the concept of the aleph, as introduced in the tale by Jorge Luis Borges. As such, the overall construction happens as from Plato to Borges.

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In regards to the DNA rosette, the pattern shown below is a representation found iterating two suitable linear maps yielding ten tips, while guided by the binary expansion of pi.

Remarkably, spokes and rings are on the right places when compared to the image on top as it appears in biochemistry textbooks projecting the double-helix, and this improbable finding, requiring the alignment of 40,000 bits of pi, makes us scratch our heads.

Page 49.

For your enjoyment and as invitation to the family day tomorrow, here is an evolution that shows the growth of a crystal.

Page 50.

And as a teaser to get you further wired on the ideas, here there is a sample of patterns found on spherical bells over three dimensions.

These are exotic higher-dimensional patterns that are part of unexplored kaleidoscopes. One can only dream what yet higher-dimensional bells may contain.

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Although these results are clearly beautiful and intriguing, there are yet more surprises when considering the ever-positive case on a two-dimensional wire.

When both d sub 1 and d sub 2 approach plus 1, we obtain a cloud wire and the Platonic ideas define yet another bell, but now centered, or concentrated, at infinity.

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With due imagination, we may see how, in a “mystical” manner, this powerful wire, maximally positive and also thick as two-dimensional space, raises it all to the clouds, filtering any kind of disorder, thorns, and dust --except for a discrete input-- into an improbable condition of plenitude without entropy, yet reflected by the melodic bell.

How not to recognize here, in a big bang in reverse, a manifestation of freedom and true mercy?

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For in the directionality of the diagram, from x into y, and in the passage from obscure dissipation to luminous conduction at infinity, we may exclaim with Saint Paul: “Where, O death, is your victory? Where, O death, is your sting?”

Page 54.

I would like to end with a poem:

The bell peals silent,
reflecting its peace,
and inside it gathers
lovely masterpiece.

Exotic pure beauty,
o splendid delight,
this limit in fullness
stores life's designs.

Such vessel contains,
alephs of all tastes,
diatoms and crystals
including DNA.

But there is a case,
reason to rejoice:
o forward selection
that raises it all.

O plus-plus election
that opens the door...

Thank you very much!