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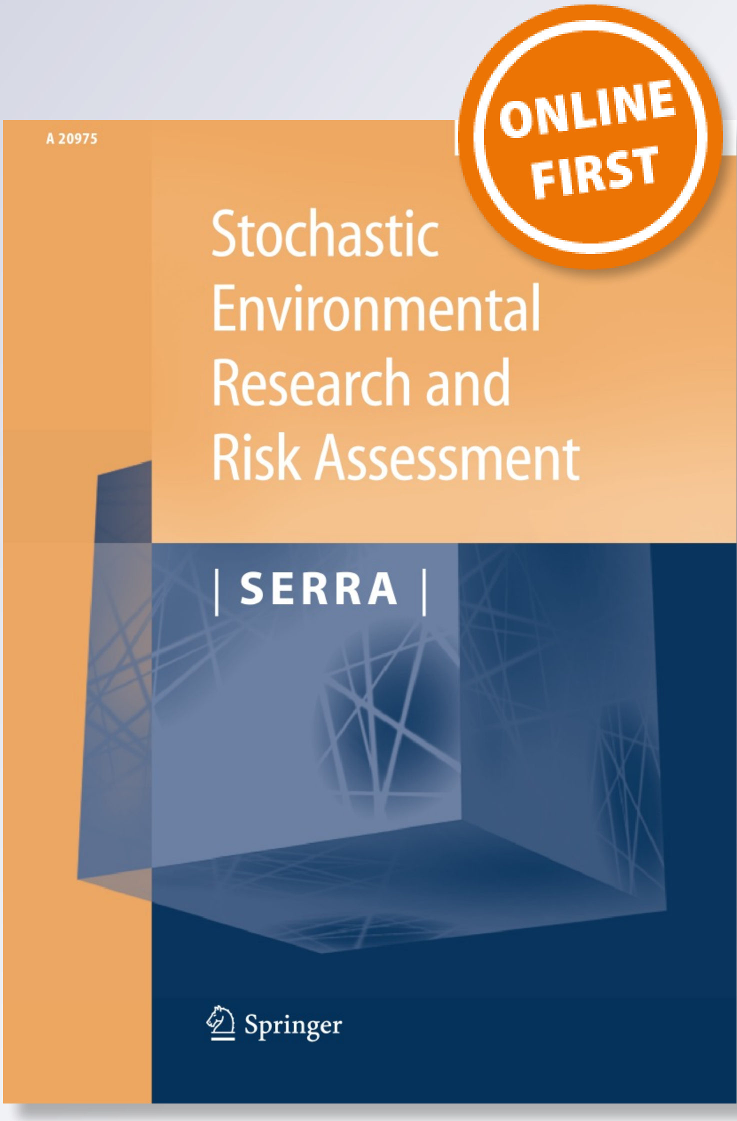

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ORIGINAL PAPER

# A physical interpretation of the deterministic fractal–multifractal method as a realization of a generalized multiplicative cascade

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Huai-Hsien Huang · Mahesh L. Maskey ·
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**Abstract** In this study, we attempt to offer a solid physical basis for the deterministic fractal–multifractal (FM) approach in geophysics (Puente, Phys Let A 161:441–447, 1992; J Hydrol 187:65–80, 1996). We show how the geometric construction of derived measures, as Platonic projections of fractal interpolating functions transforming multinomial multifractal measures, naturally defines a nontrivial cascade process that may be interpreted as a particular realization of a random multiplicative cascade. In such a light, we argue that the FM approach is as "physical" as any other phenomenological approach based on Richardson's eddies splitting, which indeed lead to well-accepted models of the intermittencies of nature, as it happens, for instance, when rainfall is interpreted as a quasi-passive tracer in a turbulent flow. Although neither a fractal interpolating function nor the specific multipliers of a random multiplicative cascade can be measured physically, we show how a fractal transformation "cuts through" plausible scenarios to produce a suitable realization that reflects specific arrangements of energies (masses) as seen in nature. This explains why the FM approach properly captures the spectrum of singularities and other statistical features of given data sets. As the FM approach faithfully encodes data sets with compression ratios typically exceeding 100:1, such a property further enhances its "physical simplicity." We also provide a connection between the FM approach and advection–diffusion processes.

**Keywords** Rainfall in time · Fractals · Multifractals · Inverse problem · Particle swarm optimization · Fractal–multifractal approach

## 1 Introduction

Modeling of natural complexity has substantially improved with recent advances in technology combined with developments of sophisticated mathematical techniques that encompass stochastic theories and fractal geometry, among others. Since natural data sets are typically erratic, noisy, intermittent, fractured, complex, or, in short, seemingly "random," it has become common to use such ideas to model them (e.g., Schertzer and Lovejoy 1987; Mandelbrot 1989; Gupta and Waymire 1993; Marsan et al. 1996; Menabde and Sivapalan 2000; Salvadori et al. 2001; Deidda et al. 2006; Sivakumar and Sharma 2008; Veneziano and Langousis 2010; Langousis et al. 2013). This has given rise to a variety of approaches, that although yielding faithful modeled sets (i.e., realizations that preserve relevant statistical and physical attributes of the records, including moments, autocorrelation, power spectrum, and multifractal spectrum), are often incapable of capturing the specific details and textures found in observed geophysical data sets, such as rainfall records.

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Encouraged by the success in defining certain deterministic fractal sets via iterations of simple maps (e.g., Barnsley 1988), Puente (1992, 1996) introduced a fractal geometric approach aimed at capturing the complexity of geophysical (hydrologic) patterns. This geometric methodology, termed as the Fractal–multifractal (FM) approach, involves deterministic derived measures obtained transforming multifractal measures defined from simple cascades via fractal interpolating functions. As has been amply demonstrated (e.g., Puente and Obregón 1996, 1999; Puente et al. 2001a, b, 2002; Obregón et al. 2002a, b; Puente and Sivakumar 2003; Puente 2004; Cortis et al. 2009; Huang et al. 2012) the geometric approach (and its extensions) produces a vast class of patterns, defined over one and higher dimensions, that have all the most salient characteristics and fine details of data sets observed in the field of geophysics, including rainfall, contaminant transport, and width functions of river networks.

Despite these encouraging outcomes, a solid physical basis of the FM approach for geophysical phenomena has continued to be elusive. In the present study, we attempt to address this issue by showing how the deterministic FM approach may be understood as a plausible realization of a general "stochastic" cascade and also as a generalization of a turbulent advection–diffusion process.

The paper is organized as follows. In Sect. 2, we describe the construction of the original fractal–multifractal approach over two dimensions. In Sect. 3, we offer a plausible physical interpretation of the FM approach (also over three dimensions) through examples of processes commonly encountered in geophysics. Finally, the concluding Sect. 4 discusses the importance and role of the FM approach in future geophysical studies.

## 2 The fractal–multifractal approach: basic concept and procedure

In its original form (Puente 1992, 1996), a fractal–multifractal (FM) pattern is obtained as the transformation of a simple multifractal measure via a fractal interpolating function. Specifically, the graph $G = \{(x, f(x)) | x \in [0,1]\}$ of such a function $f : x \to y$ passing by $N+1$ ordered points along $x$, $\{(x_n, y_n) | x_0 < \cdots < x_N, n = 0, 1, \ldots N\}$, is defined as the unique "attractor" of $N$ simple contractive affine maps (Barnsley 1988):

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}, \quad n = 1, \ldots, N, \tag{1}$$

where the vertical scalings $d_n$ satisfy $|d_n| < 1$, and the other parameters $a_n$, $c_n$, $e_n$, and $f_n$ are obtained from the following sets of initial conditions:

$$w_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \tag{2}$$

and

$$w_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \tag{3}$$

which contract the ends of the domain (over $x$) into interior subintervals.

In virtue of a fixed-point theorem for contractive maps, the attractor $G$ exists and, because of Eqs. (2) and (3), it contains the interpolating points $(x_n, y_n)$ (Barnsley 1988). Ultimately, the iteration of the simple affine mappings in Eq. (1) yields a unique (and hence deterministic) set $G$ having a fractal dimension $1 \leq D < 2$ (Barnsley 1988).

In practice, the graph of a fractal interpolating function, typically shaped as a rough "wire," may be obtained by a point-wise sampling of the attractor through a procedure known as the "chaos game" (Barnsley 1988). The idea here is to start the process at a given $(x_n, y_n)$ already in $G$ and then progressively iterate the $N$ maps $w_n$ according to, for example, the outcomes of independent "coin tosses" in order to gather more points in $G$. After a sufficiently large number of iterations, a unique invariant measure is also induced over $G$ that reflects how the attractor is filled up. The existence of such a measure (akin to a histogram in practical applications) allows computing unique—and once again, fully deterministic—projections ("shadows") over the coordinates $x$ and $y$ (denoted by $dx$ and $dy$), which turn out to display, especially over $y$, irregular and "seemingly random" shapes as found in a variety of applications (e.g., Puente 2004).

Figure 1 shows an example of a fractal function that passes by the three points $\{(0,0), (0.5,-0.35), (1,-0.2)\}$ as generated by $2^{14}$ iterations of the two maps

$$w_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ -0.51 & -0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$

and

$$w_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0.03 & -0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.5 \\ -0.35 \end{pmatrix}. \tag{5}$$

In other words, the maps $w_1$ and $w_2$ have contractive vertical scalings $d_1 = -0.8$ and $d_2 = -0.6$ and operate into the intervals $[0, 0.5]$ and $[0.5, 1]$, respectively. In addition to the graph $G$, the figure also displays the projections (histograms) $dx$ and $dy$, induced while carrying out the previously mentioned chaos game according to a biased 30:70 % proportion on $w_1$ and $w_2$, using "independent" pseudo-random tosses, starting the process from the midpoint $(0.5, -0.35)$.

As the $x$-component is independent of $y$ in Eq. (1), the implicit contractions over $x$ implied by Eqs. (2) and (3)

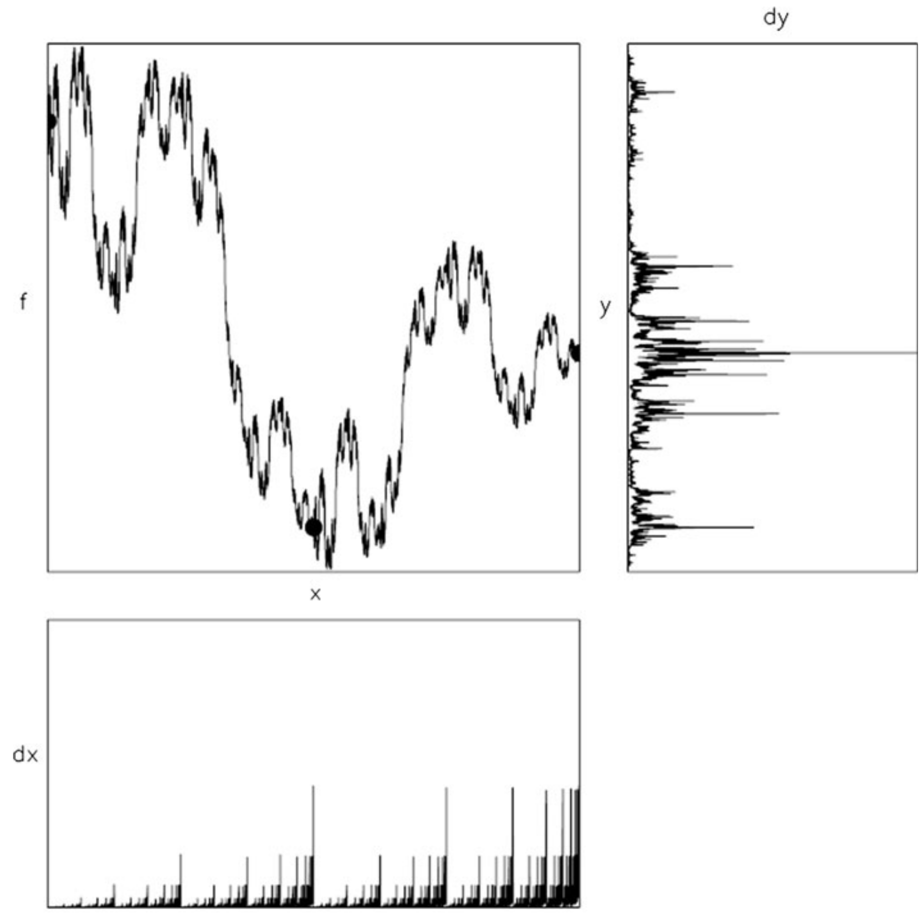



**Fig. 1** A multifractal measure *dx* is transformed via a fractal function *f* into a derived measure *dy*

result in a measure *dx* over *x* that is simply a deterministic binomial multifractal, as obtained via a classical multiplicative cascade defined over [0,1], with length scales: 0.5 and 0.5 (from the domain of the two mappings in *x*) and fixed multipliers: 0.3 and 0.7 (from the proportions used in computing the iterations), yielding a rather structured object containing an ordered sequence of spikes (e.g., Mandelbrot 1989). The connection between $f(x)$ and *dx* is the one of a function with its support: in other words, for any given $f(x)$ there exists a class of *dx* supports that are generated by different embodiments of the iterated maps, as a function of the probability of going from one map to another. While there is just one $f(x)$, it is possible to obtain a wide class of *dy*, based on the structure of the support *dx*.

The measure *dy*, being related to *dx* via the fractal function, is just the derived distribution of *dx* via *f* (e.g., Puente 1992, 2004) and such a histogram is defined, for a given value of *y*, adding the corresponding *dx*'s that satisfy $f(x) = y$. As can be seen in Fig. 1, the set *dy*, that is, the projection over *y* of the unique invariant measure generated over *G* by the chaos game, does not exhibit the same kind of repetition observed in *dx*. As may be inferred by the specific shapes of the attracting wire *f* and the "parent" measure *dx*, the implied *dy* indeed reflects the juxtaposition of such sets and results in a "seemingly random" object that, while being entirely deterministic, contains a sequence of non-trivially located spikes that look like a complex natural pattern over time (or along a line), such as a rainfall time series.

It turns out, that by varying the parameters that define a fractal interpolating function and a parent *dx*, the obtained *dy*'s populate a universe of patterns that bear resemblance to natural geophysical time series such as rainfall patterns (e.g., Obregón et al. 2002a, b; Huang et al. 2012) or wellbore log measurements and others (Puente 2004). As reflected by ubiquitous power-law scaling on their power spectra and the presence of a well-defined multifractal spectra (e.g., Puente and Obregón 1999; Puente 2004), such deterministic sets *dy* typically do not exhibit obvious trends when the dimension of the implied fractal function *f* does not approach its limiting value of two, which otherwise defines, irrespective of the input parent *dx*, Gaussian projections *dy* (e.g., Puente 1992).

As the statistics of the *dy* projections are consistent with the ones of observed data sets, this bears the question of what may be a relevant physical interpretation to be associated with the geometric FM procedure. In what follows, we shall provide specific explanations to that end.

## 3 Physical interpretation of the fractal–multifractal approach

### 3.1 The FM approach as a transformation of turbulence

In order to assign a physical interpretation to the geometric FM approach, we start by recalling that the observed energy dissipation in fully developed turbulence along a line in time or in space can be interpreted as the realization of a (progressively permuted) binomial multifractal measure with equal length scales and multipliers 0.3 and 0.7 (Meneveau and Sreenivasan 1987). When the outcome of such a multiplicative cascade is interpreted as the $dx$ input in Fig. 1, the derived measure $dy$ may be readily given an interpretation as a "transformation of turbulence," via a progressively permuted, i.e., as in $dx$, fractal interpolating function $f$ (Puente and Sivakumar 2007). Loosely speaking, as turbulence drives a quasi-passive rainfall, $dy$ may be thought of as a "reflection" or a "projection" of turbulence, with the fractal function performing the appropriate book-keeping that results in the accumulation of rain coming from consonant eddies, i.e., adding up values of $dx$ that satisfy $y = f(x)$.

When $dx$ is not the one shown in Fig. 1, that is, when such corresponds to multipliers other than 0.3 and 0.7 (or 0.7 and 0.3), the derived measure $dy$ may not be explicitly related to observations of turbulence over one dimension, but such can be thought of as a transformation of a multiplicative cascade, a more general cascade over one dimension that is consonant with the very same notion of cascades used to represent both turbulence and rainfall over three dimensions (e.g., Schertzer and Lovejoy 1987). These cases, and also those based on more than three interpolating points that lead naturally to generalized fractal interpolating functions and deterministic (multinomial) multifractal measures over $x$ (with possibly uneven length scales and various multipliers), lead to projections $dy$ that may indeed be interpreted as transformations of generalized multiplicative cascades computed via a local "integration" of the fractal function, and in a way that is akin to a weighted "fractional integration."

Among such cascading cases, there is a special condition that results yet in an alternative interpretation, and such is the situation when a uniform measure is found over $x$. Such happens, for instance, when the two multipliers are equal to 0.5 in the setting of Fig. 1, but more generally, when the sizes of intervals made by successive interpolating points over $x$ match the values of the used multipliers. In such instances, the projection $dy$ depends only on the (equally-weighted) crossings of the fractal function and such an object by itself reflects the organization of the implied rainfall, as it may also be done for turbulence records themselves (Puente and Obregón 1999). These cases, although once again fully deterministic and yielding transformations of uniformity, turn out to be consonant with the notion that rainfall may also be understood without using random cascades but rather based on nonlinearly filtering (transforming) an autoregressive process (e.g., Ferraris et al. 2003).

### 3.2 The FM approach as a generalized cascade

As previously explained, the patterns in Fig. 1 are the outcome of many iterations (on the order of $10^5$ of them) of the two maps $w_1$ and $w_2$ in Eqs. (4) and (5). By following the trail of calculations, the successive points $(x_i, y_i)$ just land on the fractal wire according to the multipliers used, and uniquely define the deterministic measures $dx$ and $dy$. While $dy$ is indeed mathematically a transformation of $dx$, one may also understand such a projection by following the construction of the fractal function and the parent multifractal over $x$, step by step.

Figure 2 shows the first few steps of the FM construction, for the same setup of Fig. 1. While the first column shows the familiar construction of a binomial multifractal following a simple multiplicative cascade, $x$ vs. $dx$, the second column displays the implied piecewise construction of the fractal interpolating function, $f : x \to y$, found by joining the original interpolating points $\{(0,0), (0.5,-0.35), (1,-0.2)\}$ with successive images of the mid-point using the mappings $w_1$ and $w_2$ according to a binary tree. While the third column portrays the "bare" derived measure, $y$ vs. $dy$, implied by just passing the corresponding mass $dx$ through the piecewise wire with points not-equally spaced over $y$, the fourth column shows a "dressed" representation of such a derived measure defined over intervals of equal size that span the minimum and the maximum of the range of the wire.

As may be appreciated in Fig. 2, the successive construction of the derived projection $dy$, either bare or dressed and clearly approaching the dressed ultimate attractor in Fig. 1 found via the chaos game, defines its own non-trivial multiplicative cascade—one that is no longer a simple binomial construct as for $dx$, but rather a complex one having distinct—seemingly random—multipliers from step to step due to the specific interplay between the parent $dx$ and the precise bending of the wire $f$.

On the light of these geometric observations, one may indeed assign an interpretation to the general $dy$ projections as realizations of "random" (conservative) multiplicative cascades implicit within the FM construction, which explains why the deterministic FM approach yields sets with well-defined multifractal spectra (or co-dimension functions) (e.g., Puente and Obregón 1999; Puente 2004; Cortis et al. 2009; Huang et al. 2012). Strikingly and as appreciated in the last two columns of Fig. 2, a

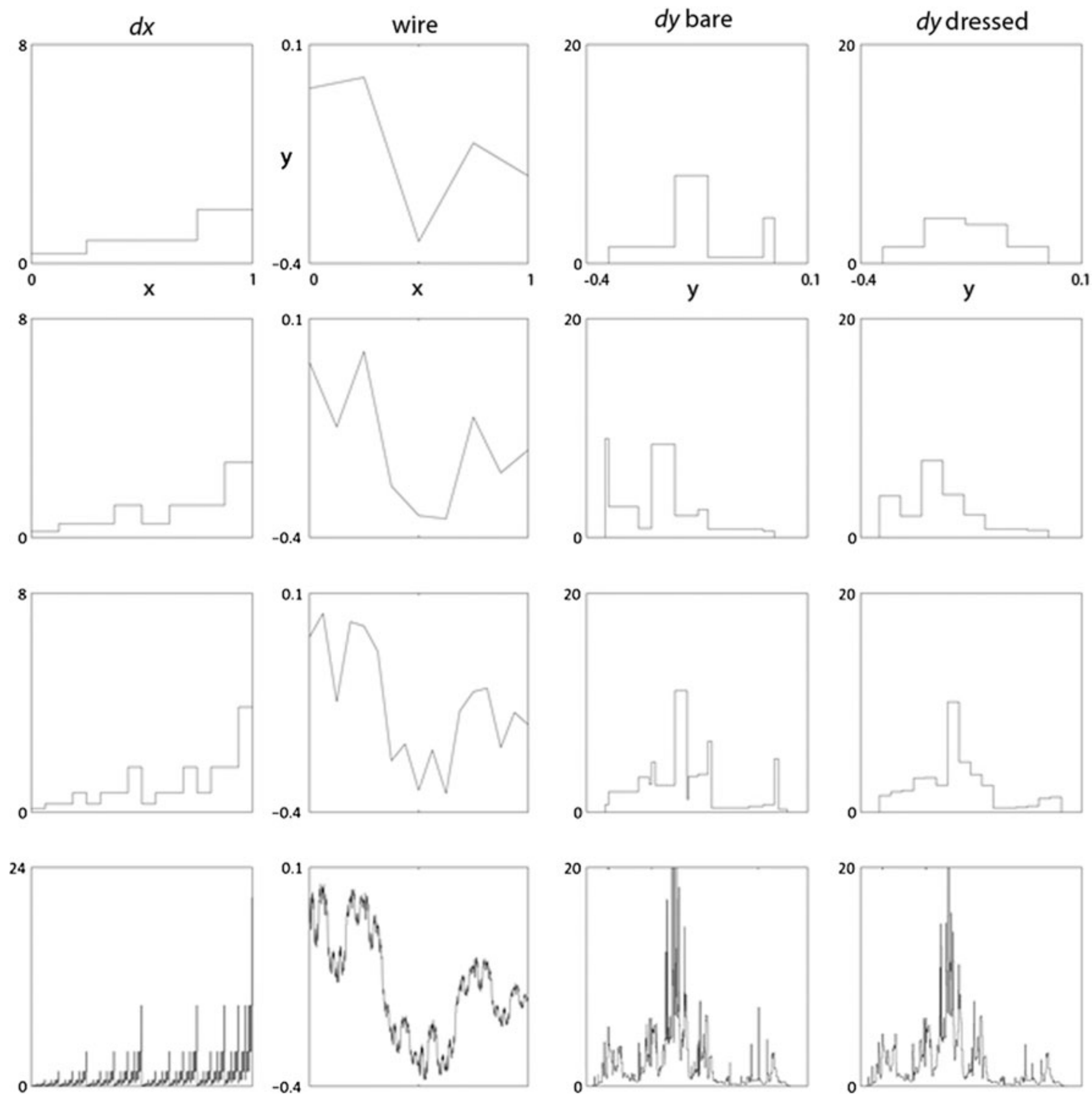



**Fig. 2** A step-by-step construction of the derived measure *dy* of Fig. 1. From *left* to *right* the binomial parent input, the fractal interpolating function, the bare derived histogram and the dressed derived histogram. The *rows* correspond to the second, third, fourth, and ninth levels of the implicit cascade, which is simple in *x* but non-trivial in *y*. While *x*-values are equally spaced, *y*-values are not as they come from images of the mid-point via the two maps $w_1$ and $w_2$

deterministic fractal transformation "cuts through" all the inherent randomness in the construction to efficiently parameterize a suitable realization that reflects specific arrangements of energies (masses) as seen in nature, an arrangement that may also be visualized in terms of the various layers of *dx*, as portrayed on Fig. 3.

The FM approach, via its implicit non-trivial cascade, thus provides a physical deterministic model of rainfall (and other processes) as a scalar driven by a general turbulent cascade reflecting Richardson's phenomenology, as has arisen, in a more general sense, using stochastic methods (e.g., Schertzer and Lovejoy 1987). While randomness describes the implicit inner structure of a FM construction, such a feature represents only a transient path towards a stable projected measure *dy*, which becomes the relevant deterministic attractor for a given realization.

As may be expected, the FM cascade process can be retraced over more than one dimension via generalizations of Eq. (1) that now include say another coordinate *z* and yield a fractal function $g : x \rightarrow (y, z)$ (e.g., Puente 2004). In addition to diagrams like those in Fig. 2, but now containing a component from *x* to *z* and an additional marginal measure over *z*, *dz*, one can also define general non-trivial cascades over two dimensions $(y, z)$ to obtain successive joint derived measures *dyz*.

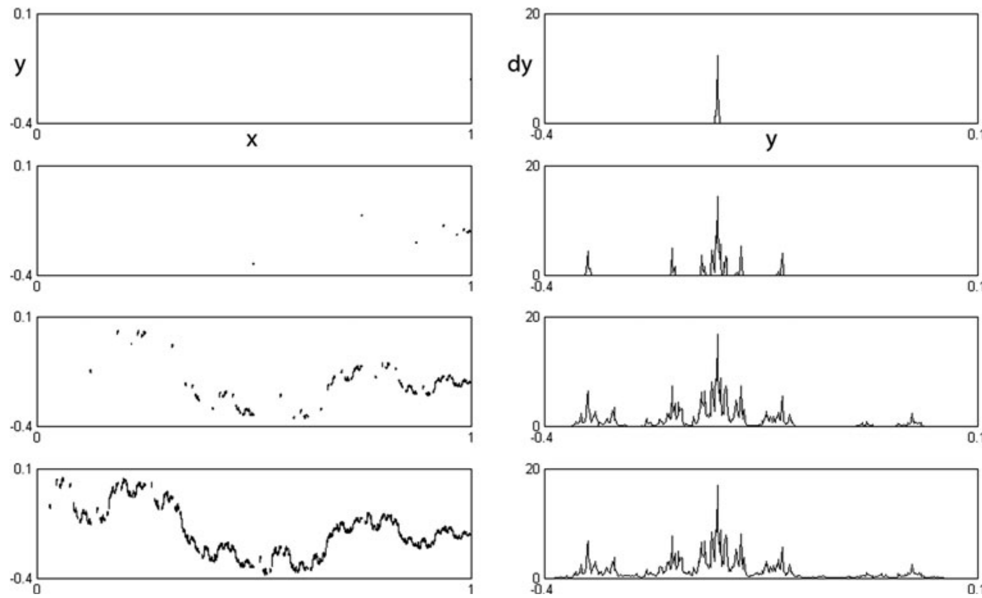



**Fig. 3** A decomposition of the derived measure in Figs. 1 and 2 in terms of layers on the parent measure $dx$. Locations of points on the attracting wire are followed by the masses they produce, from the first, second and first, fourth to first, and sixth to first layers on $dx$, as shown in Fig. 1

As illustrated in Fig. 4, the geometric construction, for dressed values computed via the specific bending of a three-dimensional wire, implicitly contains an obvious but non-trivial cascade over two dimensions that further explains why the FM approach may be used to represent and simulate sets exhibiting multifractal properties over one or higher dimensions. For even though neither the fractal function nor the level-to-level multipliers can be measured directly for any application, the FM construction is akin to the classical notion of random multiplicative cascades and provides a useful parameterization of the physical process that can further be related to physical observables.

### 3.3 The FM approach as an advection–diffusion process

The physical basis of the FM approach can also be appreciated by studying the limit of such a process in the two-dimensional case for scalings $d_1 = -d_2 \approx 1$ (see Eq. (1)). In this case, the derived measure $dy$ corresponds universally to Gaussian distributions (Puente 1992), which are the Green functions of the advection–diffusion equation (ADE):

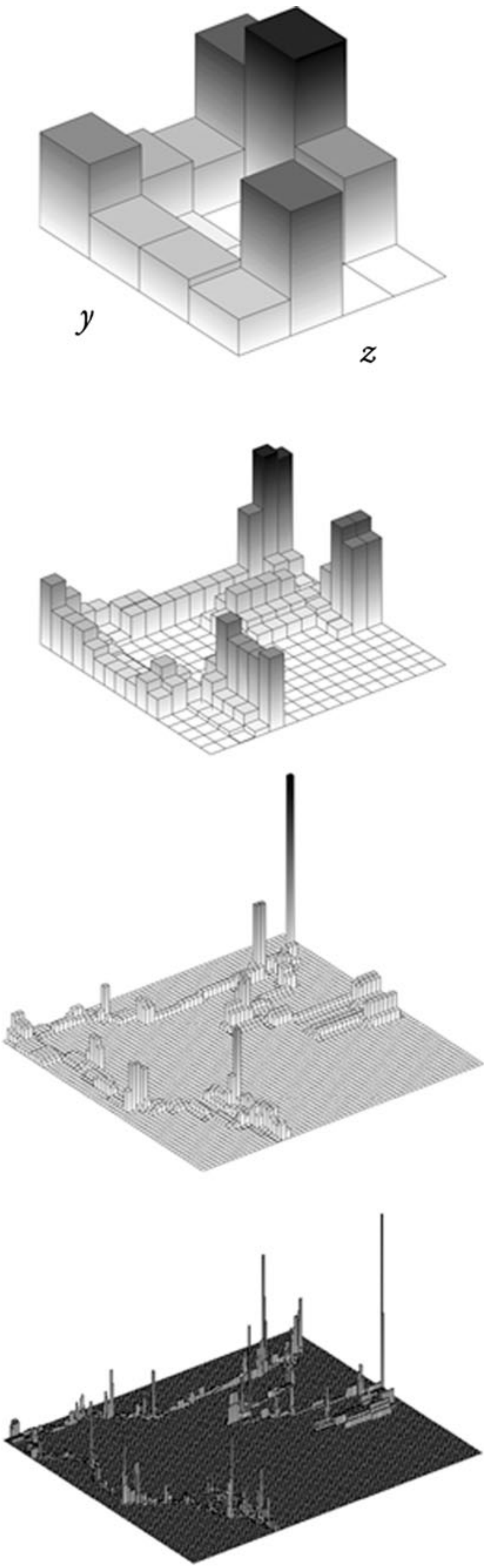
$$\frac{\partial c}{\partial t} = -\nabla \cdot v(c - \alpha \nabla c), \qquad (6)$$

where $c$ is the concentration, $v$ the velocity and $\alpha$ the diffusivity.

Figure 5 illustrates how the FM approach may be used to produce projections for fixed scalings $d_1 = -d_2 = 0.995$ (wiggly) that match the analytical solution of the ADE (smooth). In this case, there exists a simple one-to-one correspondence between the scaling parameters $d_1$ and $d_2$ of the FM approach and the time-dependent variance of the ADE distribution for the case when there is a perfectly homogeneous substrate. We note here that the physical process associated with ADE can itself be described by a random walk, i.e., a random process converging towards a stable deterministic attractor (Einstein 1905). As for the classical random walk case, large numbers of particles are moved along random trajectories in phase space, and their density at any given spatial position and time is obtained through a projection.

When the aforementioned condition of homogeneity is not satisfied, deviations from the theory appear as a non-locality of the process in time and or space (Berkowitz et al. 2006). The literature focusing on the statistical mechanics of anomalous transport is vast, and the subject is far from being completely understood. The tools of statistical mechanics used to provide physical pictures of transport in complex substrates, however, have the inherent limitation of dealing with smooth distributions of the random variables. So for instance, the random distribution of jump lengths in the classical random walk is assumed to be a smooth probability density function (pdf) with finite first and second moments (usually a Gaussian distribution itself) and generalizations of the random walk process to more general continuous time random walks (CTRW), are also based on smooth pdf for the particles retention times (Cortis 2007).

In reality, however, there are numerous physical situations where the smooth character of the transported quantities is lost and a multifractal character of the density emerges, either as a result of a short sampling time, or of an extremely complex (heterogeneous) substrate. This motivated us in




◀**Fig. 4** A step-by-step construction of a joint derived measure $dyz$ via mappings that generalize those in Eq. (1). The *rows* correspond to the second, fourth, sixth and eight levels of the implicit cascade

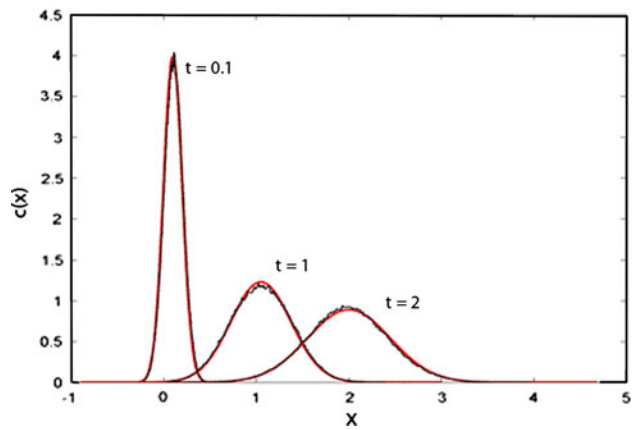



**Fig. 5** Time evolution of the breakthrough concentration C(x) for the 1D advection–diffusion Eq. (6) for a pulse at $x = 0$ at three successive times $t = 0.1$, 1.0, and 2.0 (*smooth*) and a FM fit using $d_1 = -d_2 = 0.995$ (*wiggly*)

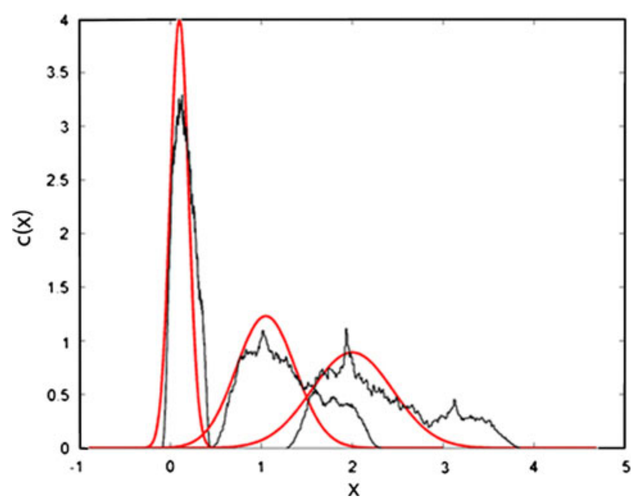



**Fig. 6** Same setting as in Fig. 5. But now the effect of changing the scalings as a function of time according to the evolutions $d_1(t) = 0.0298t + 0.7621$ and $d_2(t) = 0.0579t - 0.8958$

exploring the possibility of mimicking the non-smooth character of the implied density functions as depicted in Fig. 6, by using time-varying FM scaling parameters according to the expressions: $d_1(t) = 0.0289 \times t + 0.7621$ and $d_2(t) = 0.0579 \times t - 0.8958$. As implied by the reasonableness of the new jagged distributions, we are thus in possession of an extremely powerful deterministic tool which may allow us to relate the spatial distribution of the heterogeneities to a small number of geometric parameters, simply relying on observed time-varying spatial concentrations.

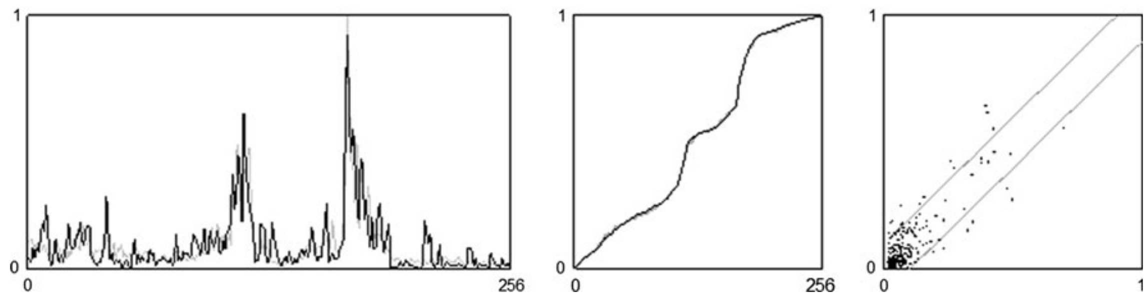



**Fig. 7** (*Left*) A storm in Iowa City having 8,192 data points (*gray*), modeled via a FM representation using 13 parameters (*black*), thus resulting in a compression of 630:1. The maximum error in cumulative distribution (*Middle*) is 1.7 %. (*Right*) Scatterplot of model and data, showing a Nash–Sutcliff coefficient of 0.71 (after Huang et al. 2012)

A further example of the FM approach to contamination issues is contained in Puente et al. (2001a, b), who successfully applied the projection ideas to study the dynamics and predict the evolution of the Borden site groundwater bromide plume using three dimensional wires driven by uniform measures. As FM representations closely preserved the geometry of patterns, it is noteworthy to point out that even if the underlying wires may not be physically measured in nature, some of their parameters may be assigned a "physical" interpretation. For instance, the coordinates that pivot the points of required fractal functions determine the center of mass and the dispersion of snap-shots of the plume and (higher dimensional extensions of) the scaling parameters $d_1$ and $d_2$ (which turn out to be matrices in polar coordinates) dictate the orientation of the plume and how closely such may be described by ellipses. While the remainder of the parameters account for the fine details of the distributions, further studies are necessary to understand their relation to the underlying heterogeneous substrate and such is a topic of ongoing research.

## 4 Why is the FM approach a good idea?

Past studies and the above discussion make it abundantly clear that the FM methodology is indeed a highly suitable tool to simulate a variety of geophysical records (e.g., rainfall) just as other (stochastic) approaches can. Compared to other methods, the FM approach also turns out to be a particularly good idea as it opens the possibility of describing, in a holistic manner, complex data sets (Cortis et al. 2009). That this is indeed the case is illustrated in Fig. 7 for a rainfall event gathered in Iowa City, USA.

As rainfall representations having rather small maximum errors in cumulative distribution (less than 2 %) may be obtained (e.g., Huang et al. 2012) and given the intrinsic errors in measuring precipitation (Lanza and Vuerich 2009), the FM fit of Fig. 7, and others for various data sets, represent faithful deterministic descriptions of the natural process. Since only a few parameters are required to define a fractal function and a parent multifractal measure (see Eq. 1), such representations compress the information with remarkable ratios that exceed 100:1, a feat that is difficult to achieve with classical stochastic techniques.

The above results and discussion for turbulence and rainfall clearly illustrate how the deterministic and geometric FM procedure yields outcomes that can be interpreted as realizations of general random multiplicative cascades. Therefore, the FM approach is as "physical" as other methods based on the phenomenological concept of stochastic cascades. The discussion on the advection–diffusion process in heterogeneous media, through an example of contaminant transport, provides further support to the utility and suitability of the FM approach for studying geophysical processes. Even though the explicit relationship between the structure of the heterogeneity and the evolution of the FM parameters remains, to date, an open problem, there is no question that parameters of the FM method may be assigned meaningful physical interpretations.

Clearly, there are several procedures that can be used to simulate the intermittencies and other properties of geophysical processes, including rainfall; however, the merit of the FM approach rests in its potential ability to fully encode a complex data set in its totality with a mere handful of parameters that provides new vistas for classifying patterns and understanding dynamics. The physical interpretation presented in this study certainly strengthens the case for the application of the FM methodology in geophysics. Therefore, we hope that the method will find even more and widespread applications in the field of geophysics and beyond.

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