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THE GAUSSIAN DISTRIBUTION REVISITED

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Abstract

A new construction of the Gaussian distribution is introduced and proven. The procedure consists of using fractal interpolating functions, with graphs having increasing fractal dimensions, to transform arbitrary continuous probability measures defined over a closed interval. Specifically, let X be any probability measure on the closed interval I with a continuous cumulative distribution. And let $f_{\Theta,D}: I \to R$ be a deterministic continuous fractal interpolating function, as introduced by Barnsley (1986), with parameters Θ and fractal dimension for its graph D. Then, the derived measure $Y = f_{\Theta,D}(X)$ tends to a Gaussian for all parameters Θ such that $D \to 2$, for all X. This result illustrates that plane-filling fractal interpolating functions are 'intrinsically Gaussian'. It explains that close approximations to the Gaussian may be obtained transforming any continuous probability measure via a single nearly-plane filling fractal interpolator.

DETERMINISTIC FRACTAL GEOMETRY; MULTIFRACTAL MEASURES; FRACTAL INTERPOLATING FUNCTIONS; GAUSSIAN DISTRIBUTION

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1. Introduction

This work provides a new construction of the Gaussian distribution as a deterministic transformation of arbitrary diffuse probability measures (that is, with continuous cumulative distribution functions). The transformations used belong to the family of deterministic continuous fractal interpolating functions, as introduced by Barnsley (1986). Specifically, it is shown that given a set of non-aligned interpolating points there exists a sequence of fractal functions $f_{\Theta,D}$, with parameters Θ and fractal dimension D for its graph, such that when $D \to 2$ they provide a derived Gaussian for any diffuse parent measure. Moreover, it is shown that the result holds for sequences based on all possible alternative parametrizations of

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functions $f_{\Theta,D}$, leading to the conclusion that plane-filling fractal interpolating functions are 'intrinsically Gaussian'.

Given that the Gaussian distribution is uniquely characterized by all its moments (see for example Billingsley (1986)), a proof of the result is given in terms of properly standardized variables. It is illustrated that the new construction leads to an orderly approximation of the Gaussian measure in the sense that accurate fittings of high order moments (even or odd) automatically lead to even increased precisions on low order moments (even or odd).

In addition to its mathematical relevance, the new construction may have physical significance. This is for two reasons. The first is the increasing importance of fractal geometry as a tool to describe nature; see for example Mandelbrot (1982) and Kaye (1989). And the second is the fact that the result holds, in a natural way, for measures that are increasingly pertinent in the study of turbulence: the family of multinomial multifractal measures; see for example Meneveau and Sreenivasan (1987), Mandelbrot (1989), and Sreenivasan (1991). Loosely speaking, the Gaussian distribution appears transforming 'turbulent' measures via plane-filling fractal functions. The Gaussian, then, could be interpreted as a 'reflection' of turbulence: and the plane-filling fractals provide the bridge between 'disorder' and 'harmony'.

The organization of this paper is as follows. First, a geometric introduction to the theory of fractal interpolating functions and multinomial multifractal measures is given, together with a brief explanation of the result shown in this work. Second, the mathematics of fractal interpolation are fully reviewed, with general expressions given for the fractal dimension D in terms of function parameters Θ. Third, the main theorems regarding the Gaussian limit are proven, concentrating on analytic formulas for the moments of the derived distributions. The result is proven first for a uniform parent measure and then it is extended to arbitrary diffuse measures. A discussion pertaining to the validity of the results via existing central limit theorems is given next. The paper concludes with a summary and remarks.

2. The geometry of fractal transformations and multifractal measures

Consider the set of three data points (equally spaced in x) as illustrated in Figure 1(a). Fix a real number $0 \leq z < 1$, draw lines between the points and locate two intermediate points going up and down the distance z from the mid-points of the line segments, as shown in Figure 1(b). Continue the process by joining the newly acquired points and the original ones, locating four additional intermediate points (in the middle of the line segments) by going up, down, and down, up respectively a distance z^2, as shown in Figure 1(c). Carry the process *ad infinitum*, adding at the nth state 2^n intermediate points with vertical displacements of magnitude z^n. Take displacements at stage n those of stage $n-1$ on the first half, and the reciprocal of these on the second half. For instance, in the third stage use as vertical

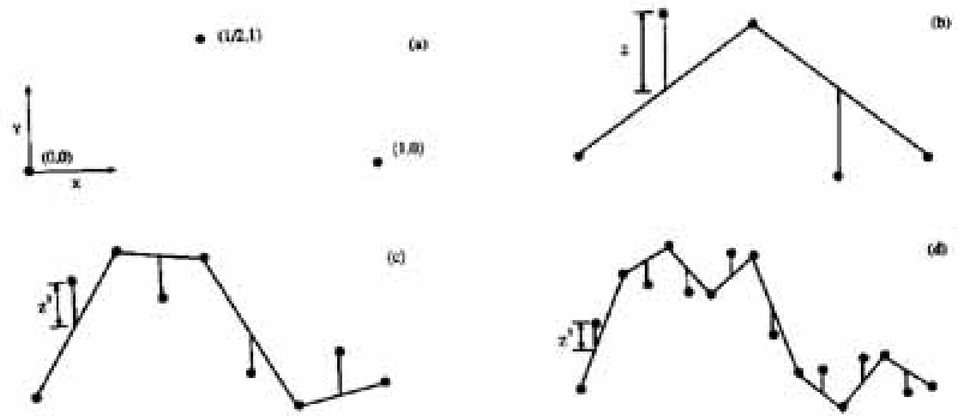

Figure 1. Recursive construction of fractal interpolating function.

displacements: up, down, down, up, and down, up, up, down, as shown in Figure 1(d).

The sequence of piecewise linear functions thus constructed converges to a continuous function $f_z:[0,1]\to R$, which, by virtue of its definition, perfectly interpolates the original three data points. Figure 2 exhibits the limiting functions for varying values of z. As shown in Figure 2(a), when z equals zero linear interpolation between the points is obtained. As seen in Figures 2(b, c and d), when z increases, the resulting function maintains the same overall appearance but becomes increasingly jagged. For $z \leqq 0{\cdot}5$, f_z may be shown to be differentiable almost everywhere with respect to the Lebesgue measure. But, as z increases beyond 0·5, f_z loses its differentiability, and its graph covers more and more space on the plane, leading to fractal (intermediate) dimensions, D, between 1 and 2; see Barnsley (1986, 1988). The resulting graphs are in all cases *self-affine*; if a restriction of any function f_z over any subinterval $[(i-1)\cdot 2^{-(n+1)}, i\cdot 2^{-(n+1)}]$, $i=1,\cdots,2^{n+1}$ at stage n is adequately enlarged (with two distinct horizontal and vertical scales), then the whole function f_z over $[0,1]$ is recovered. All functions f_z, even in the case when the fractal dimension of its graph is one, are examples of so-called *deterministic fractal interpolating functions*, as introduced by Barnsley (1986, 1988).

As any function f_z is recursively constructed, two probability measures are naturally generated by counting the relative frequencies of the acquired points in

Figure 2. Examples of fractal interpolators: (a) $z=0$, (b) $z=0{\cdot}45$, (c) $z=0{\cdot}7$, (d) $z=0{\cdot}995$

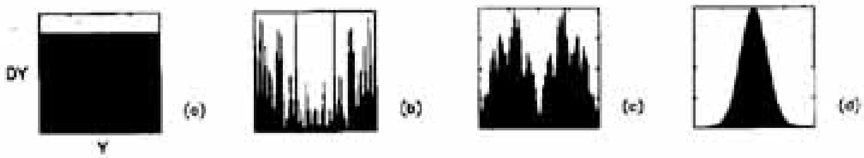

Figure 3. Derived measures $Y = f_z(U)$ for the fractal interpolators in Figure 2.

the x and y coordinates. In x, a uniform measure is obtained. This is easily seen by observing that all the acquired points up to stage n are the dyadic rationals $i \cdot 2^{-(n+1)}$, $i = 0, 1, \cdots, 2^{n+1}$. Due to the continuity of an interpolating function, measures found in y may be interpreted as being derived from the uniform measure in x via f_z, i.e. $Y = f_z(U)$, with U denoting the uniform probability measure in $[0, 1]$ and Y representing the derived measure in y that is, $Y(B) = U(f_z^{-1}(B)) = U(x : f_z(x) \in B)$ for a Borel subset B. The vertical displacement parameter z dictates the kind of measure that appears in y. Figure 3 shows the measures in y corresponding to the cases reported in Figure 2. As seen in Figure 3(a), when $z = 0$ a uniform measure is also obtained in y, as expected. As z is increased, Figures 3(b, c and d) show that non-uniform derived measures of varying shapes are obtained. As z approaches 1, the graph of f_z nearly fills up the plane and has a fractal dimension close to 2. As seen in Figure 3(d), the obtained derived measure closely resembles a Gaussian.

Notice that the domain of the measure in y, being the range of a continuous function defined over a closed interval, is finite when $z < 1$. Consequently, the graph shown in Figure 3(d) is *not* truly Gaussian. What will be shown first in this work is that a derived Gaussian is obtained in y in the limit when $z \to 1$.

As mentioned in the introduction, multifractal measures are being increasingly identified as relevant models of physical phenomena, especially in circumstances related to turbulence. Such measures also appear in the classical problem of the gambler's ruin in relation to the strategy of bold play; see for example Dubins and Savage (1968), Feller (1968), Billingsley (1983) and Feder (1988). Despite their intricate appearance, as seen on the bottom of Figure 5, they can be easily generated using a recursive procedure. Figure 4 illustrates the construction of a multiplicative binomial multifractal for the case of equal length scales. An originally uniform bar

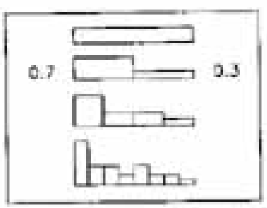

Figure 4. Geometric construction of binomial multifractal measure.

is cut by a prespecified factor p, $0<p<1$. Then, the first piece is piled-up and the second stretched so that a stair with two steps of equal lengths and masses p and $(1-p)$ is obtained. The process is repeated at each piece *ad infinitum*. At stage n of the construction there are $\binom{n}{k}$ segments of length $(\frac{1}{2})^n$ and mass $p^k \cdot (1-p)^{n-k}$, for $k=0,1,\cdots,n$, thus forming a 'layered' measure. In the limit when n tends to infinity, the number of layers also goes to infinity and the set of points that corresponds to each layer becomes a fractal subset of the interval $[0,1]$ (like the classical Cantor set). For this reason the measure obtained is termed *multifractal*; see Mandelbrot (1989). Binomial multifractals with different length scales are obtained if all mass redistributions are made over non-equal subintervals. When the original mass is partitioned into more than two pieces, general *multiplicative multinomial multifractal measures* are obtained. It will be shown later that these measures appear in a very natural way when computing a fractal interpolator following a Monte Carlo approach.

If arbitrary binomial multifractals, like the one constructed in Figure 4 for any $0<p<1$, replace the uniform distribution in x, then the same overall patterns are obtained. As illustrated in Figure 5, a binomial multifractal measure with $p=0{\cdot}7$ is transformed into an almost Gaussian in y, by employing the *same* fractal interpolating function f_z, $(z \approx 0{\cdot}995)$ as used to get an almost Gaussian distribution from a uniform measure. As will be proven in this work, the result is even more general: the measure in x may be *any* diffuse probability measure. For z close to 1, the same fractal interpolating function f_z gives Gaussian-like derived distributions in y for a very wide variety of parent measures in x.

It will also be shown that the result holds for any alternative parametrization of fractal interpolating functions, other than the exhibited f_z, as introduced by Barnsley (1986). When $D \to 2$ these functions $f_{\Theta,D}$, with parameters Θ and fractal dimension D for its graph, are 'intrinsically Gaussian': they give a derived Gaussian distribution for any diffuse parent probability measure. This *universal* result is proven based on the self-affinity of the fractal functions and the fact that the result holds for the uniform measure.

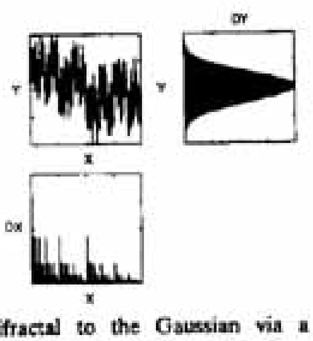

Figure 5. From a binomial multifractal to the Gaussian via a fractal interpolator. Data points $\{(0,0),(0{\cdot}5,1),(1,0)\}$; $p_1=0{\cdot}7$, $p_2=0{\cdot}3$; $d_1=-d_2=0{\cdot}995$.

3. The mathematics of fractal interpolators

Given a set of data points on the plane $\{(x_n, y_n);\ x_0 < \cdots < x_N,\ n = 0, 1, \cdots, N\}$, which from now on are supposed not to be aligned, it is possible to construct continuous functions over $I = [x_0, x_N]$, whose graphs have any prespecified fractal dimension, and which perfectly interpolate them.

Following Barnsley (1986, 1988), such an interpolator may be constructed as follows. Define N affine mappings on the plane, having the special form

$$(1)\qquad w_n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}, \qquad n = 1, \cdots, N,$$

and satisfying the conditions

$$(2)\qquad w_n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \qquad w_n\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad n = 1, \cdots, N.$$

Then, if all affine mappings w_n are contractile ($0 \le |a_n| < 1$, $0 \le |d_n| < 1$), the unique attractor set G guaranteed by fixed-point theorems, $G = w_1(G) \cup \cdots \cup w_N(G)$, gives the graph of a continuous self-affine function $f: [x_0, x_N] \to R$, which satisfies $f(x_i) = y_i$, for $i = 0, 1, \cdots, N$.

Equations (1) and (2) result in N sets of four linear equations, from which the parameters a_n, c_n, e_n and f_n may be computed in terms of the data point coordinates and the parameters d_n:

$$a_n = \frac{(x_n - x_{n-1})}{(x_N - x_0)} \qquad e_n = \frac{(x_N x_{n-1} - x_0 x_n)}{(x_N - x_0)}$$

$$c_n = \frac{(y_n - y_{n-1})}{(x_N - x_0)} - d_n \frac{(y_N - y_0)}{(x_N - x_0)} \qquad f_n = \frac{(x_N y_{n-1} - x_0 y_n)}{(x_N - x_0)} - d_n \frac{(x_N y_0 - x_0 y_N)}{(x_N - x_0)}$$

for $n = 1, 2, \cdots, N$.

The fractal dimension D of the graph G is determined in terms of the horizontal contractions a_n and the vertical scalings d_n as follows. If $\sum |d_n| > 1$, then D is the unique solution of $\sum |d_n| a_n^{D-1} = 1$, otherwise $D = 1$.

In practice, if the fractal dimension of the graph G is specified, it is easy to determine alternative sets of vertical scalings d_n which give such fractal dimension D. When the data points are equally spaced in x, the choice is particularly simple because, then, $a_n = 1/N$ for $n = 1, 2, \cdots, N$, and

$$(3)\qquad D = \max\left(1, 1 + \frac{\log \sum |d_n|}{\log N}\right).$$

Notice that the fractal dimension D tends to 2 when the absolute value of all vertical scalings tends to 1. Observe also that different parametrizations of possible functions appear considering all alternative 2^N sign combinations of vertical scalings. For instance, when $N=2$ the fractal function interpolates three data points and d_1 and d_2 could be, respectively: positive–positive, positive–negative, negative–positive, and negative–negative. In fact, the graphs f_z shown in Figures 1 and 2 are obtained taking $d_1=-d_2=z>0$ for varying values of z, considering a fixed set of interpolating points. These 2^N alternative parametrizations make up the set of functions that are considered in the interim. They are denoted $f_{\Theta,D}$, with Θ being all affine mapping parameters (the data point coordinates and the vertical scalings), and $D=D(\Theta)$ giving the fractal dimension of its graph. Whenever possible, both $f_{\Theta,D}$ and f will be used interchangeably to denote a fractal interpolating function.

An interpolation function f may be computed by means of simple algorithms. Observe that the space $C_0(I,R)$ of all continuous functions $h:I\to R$ such that $h(x_0)=y_0$ and $h(x_N)=y_N$, endowed with the supremum metric, constitutes a complete metric space. Call L_n the x-coordinate of w_n:

$$L_n : I \to I$$

$$x \to a_n x + e_n, \qquad n = 1, 2, \cdots, N,$$

and define a transformation $T:C_0(I,R)\to C_0(I,R)$ by $(Th)(x)=c_nL_n^{-1}(x)+d_nh(L_n^{-1}(x))+f_n$, $x\in[x_{n-1},x_n]$. Then, conditions set forth in the definition of the affine mappings w_n and the contractile conditions ensure that T is well defined, and that it is a contraction mapping on $C_0(I,R)$; see Barnsley (1986). It follows then that T has a unique fixed point. In fact, the fractal interpolation function f is precisely such a fixed point, and it can be computed by iterating T: $f(x)=\lim_{n\to\infty}T^n(f_0(x))$. T^n denotes the nth composite $T\circ T\circ\cdots\circ T$ (n times), and f_0 is any function in $C_0(I,R)$. For obvious reasons, f is called a *deterministic fractal interpolation function*.

A good choice for f_0 is the linear interpolation function passing by the original points. Notice that iterating such an f_0 leads precisely to the geometric procedure used previously in Figure 1 to introduce the fractal interpolators f_z, using a set of three data points and $d_1=-d_2=z$. In general, G is found following an infinite N-ary tree rooted at any observation point, say (x_1,y_1). The first N nodes are the images of the root point, obtained using the N affine mappings, and the tree continues on ad infinitum applying the affine mappings to the newly acquired points on the attractor.

Yet another way to construct G is to follow a single non-trival branch of the N-ary tree, using each mapping w_n proportionally to a prespecified N-tuple of weights $\vec{p}=(p_1,\cdots,p_N)$, where $\sum p_i=1$ and each $p_i>0$, and independently from level to level. This Monte Carlo approach, called the 'chaos game', yields a dense subset of the deterministic attractor by virtue of the law of large numbers, and provides an easy and fast algorithm to display G on a computer screen. The set of mappings $\{w_n : n = 1, 2, \cdots, N\}$ is called an *iterated function system* (IFS) by virtue of this construction of the unique set G; see Barnsley (1986, 1988).

As with the construction following all levels of the N-ary tree (i.e. as previously illustrated in Figures 3 for a binary tree), playing the chaos game also generates unique stationary measures on the x and y coordinates; see Barnsley (1986). The structure of the measures found in x is well understood. They are the multiplicative multinomial multifractal measures with parameters $p_1, \cdots, p_N$, and length scales given by the horizontal contractions a_n (i.e. the relative horizontal spacing of the data points). When one or more weights are chosen to be 0, these measures may be defined over arbitrary Cantor sets, and the corresponding distributions are 'Devil's staircases' with derivatives of the distribution functions being 0 almost everywhere with respect to the Lebesgue measure.

The chaos game also generates a measure μ on the graph G which is simply the lift of the unique stationary measure found in x. In other words, if $h : I \to G$ is the homeomorphism defined by $h(x) = (x, f(x))$ and if X is the multiplicative multinomial multifractal measure on x, then $\mu(h(B)) = X(B)$ for each Borel subset B. Puente (1992) has recently reported on the structure of the derived measures $Y = f(X) = f_{\Theta,D}(X)$ in y, defined by $Y = X(f^{-1}(B))$ for all Borel sets B, i.e. the projection of μ over the y axis. When the fractal dimension of the graph of f is close enough to 2, the derived measures become nearly Gaussian. This happens regardless of the number of data points, the initial geometry they form, the signs of the vertical scalings d_n, and the weights $\vec{p}$. In addition to proving these results, it will be shown later that as $D \to 2$ the derived distribution $Y = f_{\Theta,D}(X)$ approaches a Gaussian for *any* diffuse probability measure X.

As mentioned earlier, when $D < 2$, the range of $f_{\Theta,D}$ is a closed interval and the distribution in y is not truly Gaussian. It will be shown that the range of the fractal interpolator increases towards $(-\infty, \infty)$ as the fractal dimension D is increased towards its maximum value of 2. It will also be shown that in the limit the moments of the derived distribution approach those of the Gaussian, as illustrated before in Figure 5 for a uniform measure. The same sequence of functions $f_{\Theta,D}$ provides a derived Gaussian as $D \to 2$ for any diffuse parent measure in x.

4. From a uniform to a Gaussian via fractal interpolators

Even though fractal interpolating functions are defined implicitly, their moment integrals

$$f_{l,m} = \int_I x^m f^l(x)\, dx \tag{4}$$

can be evaluated explicitly; see Barnsley (1986). This may be done by employing the transformation T previously defined, and using the fact that $Tf = f$, as follows:

$$\begin{aligned} f_{l,m} &= \int_I x^m f^l(x)\,dx \\ &= \int_I x^m (Tf)^l(x)\,dx \\ &= \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} x^m (Tf)^l(x)\,dx \\ &= \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} x^m (c_n L_n^{-1}(x) + d_n f(L_n^{-1}(x)) + f_n)^l\,dx. \end{aligned}$$

Making the change of variables $y = L_n^{-1}(x)$, yields $f_{l,m} = \sum_{n=1}^{N} \int_I (a_n y + e_n)^m (c_n y + d_n f(y) + f_n)^l a_n\,dy$, which includes the term $f_{l,m}$ on both sides of the equation.

Hence $f_{l,m}$ may be solved in terms of $f_{l,i}: 0 \leq i < m$ and $f_{i,k}: 0 \leq i < m;\ i + k \leq l + m$ recursively, and consequently in terms of the data point coordinates and the affine mapping parameters. After some algebraic manipulations, these moments can be written as

$$(5)\quad f_{l,m} = \begin{cases} \dfrac{x_N^{m+1} - x_0^{m+1}}{m+1} \quad \text{if} \quad l = 0, \text{ else} \\ \dfrac{1}{1 - \sum_{n=1}^{N} a_n^{m+1} d_n^l} \Bigg[\sum_{j=0}^{m-1} f_{l,j} \Big\{ \sum_{n=1}^{N} \binom{m}{j} a_n^{j+1} d_n^l e_n^{m-j} \Big\} + \\ \sum_{p=0}^{l-1} \sum_{j=0}^{l+m-p} f_{p,j} \Big\{ \binom{l}{p} \sum_{n=1}^{N} a_n d_n^p \Big(\sum_{\substack{s+t=j \\ 0 \leq s \leq m \\ 0 \leq t \leq l-p}} \binom{m}{s} \binom{l-p}{t} a_n^s e_n^{m-s} c_n^t f_n^{l-p-t} \Big) \Big\} \Bigg]. \end{cases}$$

When $m = 0$ and $l > 0$, this formula reduces to

$$(6)\qquad f_{l,0} = \frac{\sum_{p=0}^{l-1} \sum_{j=0}^{l-p} f_{p,j} \binom{l}{p} \Big[\sum_{n=1}^{N} a_n d_n^p \binom{l-p}{j} c_n^j f_n^{l-p-j} \Big]}{1 - \sum_{n=1}^{N} a_n d_n^l}.$$

If the measure in x is uniform, U, then the moments of the derived measure $Y = f(U)$ may be computed in terms of the moments of the interpolating function f, as follows:

$$(7)\qquad m_Y^q = \int y^q\,dF_Y = \int_I f^q(x)\,dF_X = \frac{1}{x_N - x_0} \int_I f^q(x)\,dx = \frac{1}{x_N - x_0} f_{q,0}.$$

Due to the extremely cumbersome calculations involved, the following theorems deal with the particular case of three data points. First, it is shown that a Gaussian distribution is obtained as the limit of derived measures $Y = f_{\Theta,D}(U)$ when $D \to 2$. Specifically, interest centers on the convergence, in distribution, of the random variables

(8) $$g_i(X) = \frac{f_{\Theta_i,D(\Theta_i)}(X) - E(f_{\Theta_i,D(\Theta_i)}(X))}{\sqrt{V[f_{\Theta_i,D(\Theta_i)}(X)]}}$$

such that $D(\Theta_i) \to 2$ when $i \to \infty$, and X is a uniform random variable in (x_0, x_N).

Theorem 1. Let $f_{\Theta,D}$ be a fractal interpolation function with parameters Θ and fractal dimension $D(\theta)$ for its graph. Suppose that it interpolates an arbitrary set of three data points $\{(x_n, y_n) : n = 0, 1, 2\}$, and that it has a prespecified choice (signs) of scaling parameters d_1 and d_2. Consider the standardized random variables $g_i(X)$ defined above, where X is a uniform random variable over (x_0, x_2). Then these random variables converge in distribution to a Gaussian variable Z with mean 0 and variance 1, i.e. $\lim_{i \to \infty} g_i(X) \xrightarrow{\mathcal{D}} Z$, provided that $D(\Theta_i) \to 2$ when $i \to \infty$.

Note. Because the Gaussian distribution is characterized by its moments, e.g. see Billingsley (1986), the method of proof is to show that $E(g_i(X)^m) \to E(Z^m)$ for every m, i.e. zero for all odd moments and $(m-1)!! = 1 . 3 . \cdots . (m-1)$ if m is even. Attention is focused on the case of equally spaced points in x. The general case with non-equally spaced points may be proved following exactly the same reasoning. This case is not included here to avoid more complications than are necessary. Also, it is assumed that the parameters d_1 and d_2 have opposite signs. The proof for the cases when d_1 and d_2 have the same sign is quite tedious due to the absence of simplifications, and it is not given here. Recall that in lieu of Equation (3), $D = D(\Theta) \to 2 \Leftrightarrow |d_1| \to 1$ and $|d_2| \to 1$.

The proof is divided into three parts. First, relevant algebraic properties of the moments of fractal interpolators are revealed and proven. Then, the theorem is proven by checking the odd and even moments, respectively.

4.1. *Algebraic properties of moments of fractal interpolators.* Without any loss of generality, set $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, and $y_0 = 0$. Notice that all moment integrals $f_{i,m}$ depend continuously on the parameters d_n, and recall that the fractal dimension D of the graph of f tends to 2 when both $|d_1|$ and $|d_2|$ tend to 1 (see (3)). As interest centers on the case when $D \to 2$, it will be assumed that $d_1 = -d_2 = z > 0$, with z in the vicinity of 1 (exchanging signs on d_1 and d_2 gives an entirely symmetric case).

According to Equations (1) and (2) and under the preceding assumptions, the affine mappings which generate the graph of f are:

$$w_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ y_1 - y_2 z & z \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{9}$$

$$w_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -y_1 + y_2(z+1) & -z \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ y_1 \end{pmatrix}. \tag{10}$$

Let us study the general form of the expression for the moment $f_{l,m}$. Note from (5) that it is a rational function of z whose denominator is of the form $\prod_{t=1}^{r}(1 - \sum a_n^{i_t} d_n^{j_t})^{k_t}$, where i_t, j_t, and k_t, are non-negative integers for $t = 1, \cdots, r$. In the present case, this product always reduces to expressions of the form $\prod_{t=1}^{r}(1 - \sum (\frac{1}{2})^{i_t} z^{j_t})^{k_t}$. For reasons that will become clear later, the factors in the last product having the special form $(1 - z^2)^k$, $k \geqq 1$ (i.e. the terms whose roots are $+1$ and -1), are of particular importance. The letter κ will be reserved to denote the exponent of $(1 - z^2)$ in the denominator of a given moment $f_{l,m}$. A moment $f_{l,m}$ is said to be more complex than a moment $f_{i,k}$ if the former is expressed in terms of the latter (see (5)). To calculate κ, list the moments $f_{l,m}$ in order of increasing complexity as follows.

Begin with $f_{1,0}$. Substituting the parameters in (6), it is easily seen that $f_{1,0} = \frac{1}{2}y_1 + \frac{1}{4}y_2$ and consequently $\kappa(f_{1,0}) = 0$. Note that our choice of parameters implies that the factors $(1 - \sum a_n^{m+l} d_n^l) = 1$ for all odd indexes l. Then, it follows that the denominator of $f_{1,1}$ is 1, or $\kappa(f_{1,1}) = 0$. The factor $(1 - z^2)$ appears for the first time in the denominator of $f_{2,0}$, giving $\kappa(f_{2,0}) = 1$. This may be seen from (6) noticing that $f_{2,0}(z) = (8y_1y_2 - 12y_1^2z^2 + 6y_2^2z - 7y_2^2z^2 - 6y_2^2z^3 - 12y_1y_2z - 12y_1y_2z^2 + 12y_1y_2z^3 + 16y_1^2 + 8y_2^2)/[48(1 - z^2)]$.

From then on, all moments $f_{l,m}$ with $l \geqq 2$ being more complex than $f_{2,0}$ will include the factor $(1 - z^2)$ in its denominator at least once, or equivalently, $\kappa(f_{l,m}) \geqq 1$ for $l \geqq 2$. Observe that the factor $(1 - z^2)^2$ appears for the first time in the denominator of $f_{4,0}$. This is easily seen by noticing that the moments $f_{1,2}$, $f_{2,1}$, $f_{3,0}$, $f_{1,3}$, $f_{2,2}$, and $f_{3,1}$ (all needed to compute $f_{4,0}$), have as denominators, in order, 1, $(1 - z^2)(1 - (\frac{1}{2})z^2)$, $(1 - z^2)(1 - (\frac{1}{2})z^2)$, 1, $(1 - z^2)(1 - (\frac{1}{2})z^2)(1 - (\frac{1}{2})^2z^2)$, and $(1 - z^2)(1 - (\frac{1}{2})z^2)(1 - (\frac{1}{2})^2z^2)$. Consequently, the denominator of $f_{4,0}$ becomes

$$(1 - z^2)(1 - (\tfrac{1}{2})z^2)(1 - (\tfrac{1}{2})^2z^2)(1 - z^4) = (1 - z^2)^2(1 - (\tfrac{1}{2})z^2)(1 - (\tfrac{1}{2})^2z^2)(1 + z^2),$$

giving $\kappa(f_{4,0} = 2)$.

In general, only the terms of the form $f_{2t,j}$ appearing in the expansion of $f_{l,m}$ contribute to its denominator, and only the terms of the form $f_{2t,0}$, whose denominator contains the factor $(1 - z^{2t}) = (1 - z^2)(1 + z^2 + \cdots + z^{2t-2})$, contribute to increasing the exponent k. These results are summarized in the following lemma.

Lemma 1. Denote by $\kappa(f_{l,m})$ the maximum exponent k such that $(1 - z^2)^k$ is part of the denominator of the moment $f_{l,m}$. Then, $\kappa(f_{l,m}) \leqq t - 1$ if $l < 2t$, and $\kappa(f_{2t,0}) = t$.

Proof. Given the recursive nature of the moments, this lemma is easily proved by induction. The results have already been checked when $t=1$. Notice that the moment $f_{2t,0}$ can be expressed in terms of $f_{i,j}: i=0,\cdots,2t-1;\ i+j\leqq 2t$. By the induction hypothesis ($i<2t$), these moments have $\kappa(f_{i,j})\leqq t-1$, and $\kappa(f_{2(t-1),0})=t-1$, giving $\kappa(f_{2t,0})\leqq t$. But the denominator of $f_{2t,0}$ contains the factor $(1-z^{2t})$ and thus $\kappa(f_{2t,0})=t$.

Similarly, the moments $f_{l,m}$ may be expressed in terms of $\{f_{l,j}: 0\leqq j\leqq m-1\}\cup\{f_{i,j}: 0\leqq i\leqq l-1,\ i+j\leqq l+m\}$. Suppose that $l<2t$, then by the induction hypothesis, $\kappa(f_{l,j})\leqq t-1$ for $0\leqq j\leqq m-1$ and $\kappa(f_{i,j})\leqq t-1$, for $0\leqq i\leqq l-1$, $i+j\leqq l+m$. Then, by the result just proven for $\kappa(f_{2t,0})$, $\kappa(f_{l,m})\leqq t$ if $l<2(t+1)$.

4.2. *Proof of Gaussian result for odd moments.* The proven algebraic properties of the moments of fractal interpolators are used next to show that all odd moments of the standardized variables $g_i(X)$ vanish when $D(\Theta_i)\to 2$. Because $D(\Theta_i)\to 2$ is equivalent to $z\to 1$, one may write $Y(z)=f_{\Theta,D}$ and look for the moments of $Y(z)$ as $z\to 1$.

The mean and the variance of the measure $Y(z)$ are readily computed from (6). They are

$$m_Y(z)=\frac{f_{1,0}}{x_2-x_0}=f_{1,0}=\frac{y_1}{2}+\frac{y_2}{4} \tag{11}$$

and

$$VY(z)=\sigma_Y^2=f_{2,0}-(f_{1,0})^2=\frac{t(z)}{48(1-z^2)}, \tag{12}$$

where

$$t(z)=-4y_1y_2+6y_2^2z-4y_2^2z^2-6y_2^2z^3-12y_1y_2z+12y_1y_2z^3+4y_1^2+5y_2^2. \tag{13}$$

Notice that $t(1)=t(-1)=(2y_1^2-y_2)^2$. Consequently, 1 and -1 are not roots of $t(z)$ because this would contradict the fact that the original three data points $\{(0,0),(0{\cdot}5,y_1),(1,y_2)\}$ are not aligned.

The $(2m+1)$th moment of the standardized variables $g_i(X)$ simply yields

$$\begin{aligned} E[g_i(X)^{2m+1}]=\mathcal{M}(z)^{2m+1}&=\frac{M_Y(z)^{2m+1}}{\sigma_Y(z)^{2m+1}}\\ &=\frac{\sum_{q=0}^{2m+1}\binom{2m+1}{q}f_{q,0}(z)(-m_Y(z))^{2m+1-q}}{VY(z)^{m+\frac{1}{2}}} \end{aligned} \tag{14}$$

which can be written as $\mathcal{M}(z)^{2m+1}=\sum_{q=0}^{2m+1}(s_q/t(z))f_{q,0}(z)(1-z^2)^{m+\frac{1}{2}}$, where s_q is a constant, $q=1,\cdots,2m+1$. Since for each term in the sum, $q\leqq 2m+1$, Lemma 1 implies that the number of times in which 1 and -1 are roots of their denominators is at most m, i.e. $\kappa(f_{q,0}) \leqq m$. Consequently,

$$\mathcal{M}(z)^{2m+1} = (1-z^2)^{\frac{1}{2}} \cdot p(z) \tag{15}$$

where $p(z)$ is a sum of rational expressions in z whose denominators do not contain 1 and -1 as roots. This implies $\lim_{z\to 1} \mathcal{M}(z)^{2m+1} = 0$, $m \geqq 0$.

4.3. *Proof of Gaussian result for even moments.* The proof for the even moments also relies on the shown properties of the moments of fractal interpolators. For this case, the $2m$th standardized centered moment ($m \geqq 1$) is

$$\begin{aligned} E[g_z(X)^{2m}] = \mathcal{M}(z)^{2m} &= \frac{M_Y(z)^{2m}}{\sigma_Y(z)^{2m}} = \frac{\sum_{q=0}^{2m} \binom{2m}{q} f_{q,0}(z)(-m_Y(z)^{2m-q})}{VY(z)^m} \\ &= \sum_{q=0}^{2m} \frac{s_q}{t(z)} f_{q,0}(z)(1-z^2)^m \end{aligned} \tag{16}$$

where, as before, each s_q is a constant, and $t(z)$ is the numerator of $VY(z)$ which does not contain 1 and -1 as its roots.

Because of Lemma 1, all terms with $q < 2m$ give $\kappa(f_{q,0}) < m$, and therefore vanish as $z \to 1$. Consequently, the $2m$th moment is approximated by the last term of the sum, namely $\mathcal{M}(z)^{2m} = M_Y(z)^{2m}/\sigma_Y(z)^{2m} \approx f_{2m,0}(z)/VY(z)^m$, $z \approx 1$.

It is shown by induction on m that the correct even moment values are obtained in the limit when $z \to 1$. Observe that when $m = 1$ the result $\mathcal{M}(1)^2 = 1$ holds by virtue of the definitions of the mean and the variance. Shown next is a recursive expression that allows the transition from moment $2m$ to moment $2(m+1)$.

Using (6), it is easily seen that

$$\mathcal{M}(z)^{2(m+1)} \approx \frac{f_{2m+2,0}(z)}{VY(z)^{m+1}} = \frac{\sum_{p=0}^{2m+1} \sum_{j=0}^{2m+2-p} s_{p,j}(z) f_{p,j}(z)}{(1-z^{2m+2})VY(z)^{m+1}}$$

where $s_{p,j}(z)$ is a polynomial in z. Expanding the denominator gives

$$\frac{f_{2m+2,0}(z)}{VY(z)^{m+1}} = 48^{m+1}(1-z^2)^m \sum_{p=0}^{2m+1} \sum_{j=0}^{2m+2-p} \frac{s_{p,j}(z)}{t^{m+1}(z)(1+z^2+\cdots+z^{2m})} f_{p,j}(z)$$

and the term $t^{m+1}(z)(1+z^2+\cdots+z^{2m})$ does not contain 1 and -1 as roots. By using Lemma 1 and looking at the denominators of the terms $f_{p,j}(z)$, it is seen that the only non-vanishing terms when $z \to 1$ are those with $p \geqq 2m$. Therefore, if $z \approx 1$,

$$\begin{aligned} \mathcal{M}(z)^{2(m+1)} \approx \frac{1}{VY(z)^{m+1}}(1-z^{2m+2})\{&s_{2m,0}f_{2m,0}(z) + s_{2m,1}f_{2m,1}(z) \\ &+ s_{2m,2}f_{2m,2}(z) + s_{2m+1,0}f_{2m+1,0}(z) + s_{2m+1,1}f_{2m+1,1}(z)\}. \end{aligned} \tag{17}$$

As shown in Appendix A, Lemma 1 could be applied to obtain the four rational functions $f_{2m,1}$, $f_{2m,2}$, $f_{2m+1,0}$ and $f_{2m+1,1}$ in terms of $f_{2m,0}$, so that at the end $\mathcal{M}(z)^{2(m+1)}$ is written in terms of $\mathcal{M}(z)^{2m}$ when z is in the vicinity of 1. The appendix also shows a summary of the calculations needed to prove that if $\lim_{z\to 1}\mathcal{M}(z)^{2m} = (2m-1)!!$ then $\lim_{z\to 1}\mathcal{M}^{2(m+1)} = (2m+1)!!$

Remarks. As previously explained, there are two constructions of a fractal interpolating function which lead in a natural way to measures over the x coordinate. First, the following of all nodes of an N-ary tree that successively computes intermediate interpolating points. Second, the following of *one* branch of the N-ary tree being selected via a set of weights p_n, $n = 1, \cdots, N$, i.e. the chaos game. When the original data points are equally spaced in x, the first procedure induces a uniform over x. This *same* uniform is also obtained by the second method if $p_n = 1/N$, $n = 1, \cdots, N$.

When data points are not equally spaced in x, the first procedure does not generate a uniform any more, but the second construction does if the weights preserve the spacings between points, i.e. when $p_n = a_n$ for $n = 1, \cdots, N$. Therefore, the Gaussian result under non-equally spaced points in x may be thought of as coming from uneven iterations of the corresponding affine mappings w_n, i.e. Equations (1).

The recursive expressions for the moments of derived measures (Equations (5) and (6)) can be easily programmed to check the general validity of the limiting Gaussian. By selecting alternative interpolating data sets of varying sizes and alternative sign combinations on the vertical scaling parameters d_n, the reader may verify (be aware of the machine's precision!) that the derived moments approximate those of a Gaussian with increasing precision as all $|d_n| \to 1$. As an example, Table 1 shows the first ten centered standardized moments of the derived measure in y obtained while interpolating $\{(0, 0{\cdot}25), (0{\cdot}7, -0{\cdot}5), (1, 1)\}$ with $d_1 = -d_2 = z > 0$. Note that as z increases towards 1, the moments of the derived measure indeed tend to those of the standard Gaussian.

TABLE 1
Standardized centered moments in y. $\{(0, 0{\cdot}25), (0{\cdot}7, -0{\cdot}5), (1, 1)\}$. $d_1 = -d_2 = z > 0$

	Derived measure				
Order	$z = 0{\cdot}9999$	$z = 0{\cdot}99999$	$z = 0{\cdot}999999$	$z = 0{\cdot}9999999$	$\mathcal{N}(0, 1)$
3	1·9626e-5	6·2117e-7	1·9645e-8	6·2123e-10	0·0
4	2·9995556	2·9999557	2·9999956	2·99999956	3·0
5	1·9609e-4	6·2111e-6	1·9645e-7	6·21227e-9	0·0
6	14·561148	14·956016	14·995601	14·9995600	15·0
7	2·0074e-3	6·5053e-5	2·0621e-6	6·52271e-8	0·0
8	101·87414	104·68738	104·96874	104·996874	105·0
9	2·3943e-2	7·8018e-4	2·4744e-5	7·82721e-7	0·0
10	916·32349	942·13073	944·71306	944·971305	945·0

Table 1 also implies that the convergence to a Gaussian is achieved in an orderly fashion. If z is chosen close to 1, so that a given (odd or even) moment is fitted within a given acccuracy, then all (odd or even) moments of lower orders are approximated with even higher precision. The following conjecture, supported by Table 1 and many other examples, shows how a numerical remainder is shifted as $z = 1 - 10^{-n} = 0{\cdot}99\cdots$ is composed by an increasing number of nines.

Conjecture. For $n \gg 1$, $|f_{2m,0}(1-10^{-n}) - (2m-1)!!| \approx c_m/10^n$, $m \geqq 2$, $c_m < c_p$ if $m < p$ and $|f_{2m+1,0}(1-10^{-n})| \approx d_{m,\mathrm{mod}(n,2)}/10^n$, $m \geqq 1$, $d_{m,\mathrm{mod}(n,2)} < d_{p,\mathrm{mod}(n,2)}$ if $m < p$.

Note. The existence of the constants c_m, $d_{m,0}$, $d_{m,1}$ may be readily verified by the reader, employing the computer program of Equations (5). The results of Theorem 1 and the last conjecture hold also for any arbitrary setup and any number of data points.

5. From diffuse measures to Gaussians via fractal interpolators

It is shown next that the Gaussian result holds when the uniform measure in x is replaced by an arbitrary diffuse probability measure. A measure with a continuous cumulative distribution function is required so that increasingly accurate approximations could be obtained in terms of measures which are constant (uniform) over subintervals of increasingly small sizes. The repeated nature (self-affinity) of the fractal interpolating functions and the already proven Gaussian result for uniform measures are key in the proof of the following theorem.

Theorem 2. Let X be a diffuse probability measure defined on the interval $I = [x_0, x_2]$. Suppose that $f_{\Theta,D(\Theta)}$ is a fractal interpolating function which passes by the data points $\{(x_n, y_n): n = 0, 1, 2\}$, and which has a prespecified choice (signs) of scaling parameters d_1 and d_2. Consider the standardized derived measures $g_i(X)$, as defined in (8), such that $D(\theta_i) \to 2$ when $i \to \infty$. Then, $\lim_{i\to\infty} g_i(X) \xrightarrow{\mathscr{D}} Z$, where Z is a Gaussian random variable with mean 0 and variance 1.

Note. As in Theorem 1, the proof will be given for equally spaced data points in $I = [0, 1]$ and for the case when $d_1 = -d_2 = z > 0$. The proof is divided into three sections. First, it is shown that restrictions of fractal interpolating functions over dyadic subintervals $I_j = [(j-1)2^{-(n+1)}, j2^{-(n+1)}]$, $j = 1, \cdots, 2^{n+1}$ are themselves fractal interpolating functions defined over I_j, for any level n. Second, it is proven, via Theorem 1, that all restrictions of a fractal interpolating function $f_{\Theta,D}$, $Y_j(z) = f_{\Theta,D,j}: I_j \to R$, $j = 1, \cdots, 2^{n+1}$, give derived Gaussians, which have approximately the same variance as $Y(z) = f_{\Theta,D}(U)$ when $D \to 2$ $(z \to 1)$, with U being the uniform measure over $[x_0, x_2]$. Third, it is shown that weighing the Gaussian measures $Y_j(z)$ by the weights provided by a diffuse measure X also gives a

Gaussian measure. It is important to recall that the last step requires not the sum of random variables but rather the weighted sum of the distributions of those variables.

5.1. *Restrictions of fractal interpolators to dyadic subintervals.*

Lemma 2. Let f be a fractal interpolating function with vertical scalings d_n which passes through the data points $\{(x_n, y_n): n = 0, 1, 2\}$. Then, the restriction of f to any dyadic subinterval is itself a fractal interpolating function which has as parameters (a) the same vertical scalings of f, and (b) data points corresponding to the images by f of the initial, middle and end points of the dyadic interval.

Proof. Suppose that the points marked with dots in Figure 6 are the original data points and that the vertical scaling parameters of f are $d_1 = -d_2 = z > 0$. Consider the affine mappings w_1 and w_2 whose unique attractor is the graph of f (see Equation (1)). Due to the affinity of these functions and their special form, it is easy to see that they map parallelograms which contain vertical lines into parallelograms with

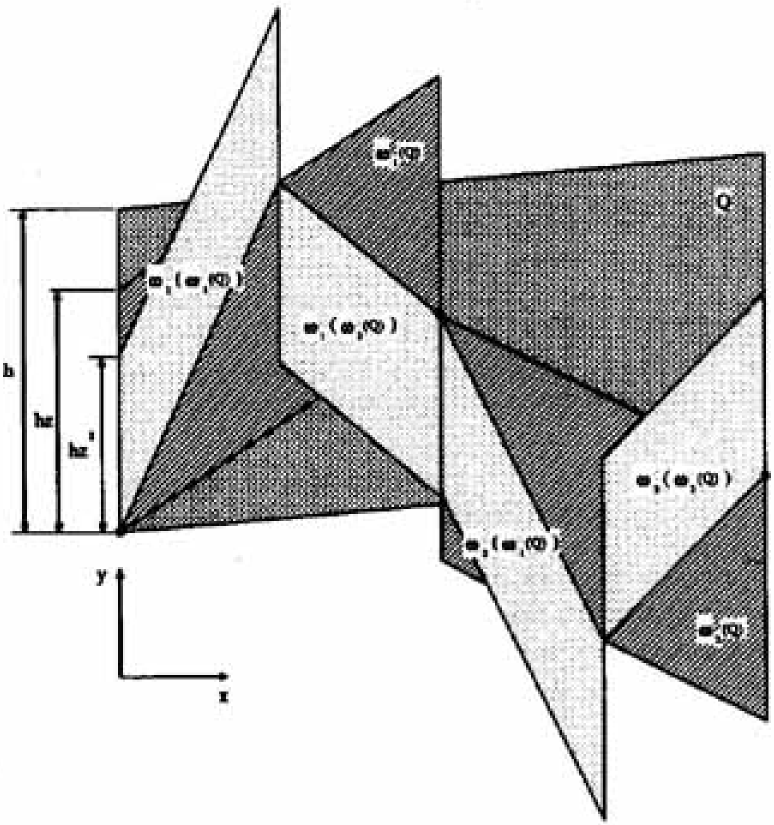

Figure 6. Schematic construction of restrictions of fractal interpolator.

vertical lines. As shown in Figure 6, w_1 maps the parallelogram Q, of height h and sides parallel to the y axis and the line that passes through (x_0, y_0) and (x_2, y_2), into a parallelogram with vertical sides and height hz. Similarly, w_2 gives another parallelogram on the right hand side but with the orientation of the vertical lines inverted. This process is repeated naturally over dyadic subintervals of $I = [x_0, x_2]$ by successive applications of the affine mappings.

Now consider the fractal interpolation function f_1 which passes through the left side of the fractal, i.e. $\{(x_0, y_0), ((x_0 + x_1)/2, f((x_0 + x_1)/2)), (x_1, y_1)\}$, and which has as scaling parameters $d_1 = -d_2 = z$. Let $\tilde{w}_1$ and $\tilde{w}_2$ be the two affine mappings which are needed to define f_1 (see (1)). Then, it is clear that $\tilde{w}_1$ and $\tilde{w}_2$ are necessarily distinct from w_1 and w_2. However, being affine, $\tilde{w}_1$ maps the parallelogram $R = w_1(Q)$ of height hz, with sides parallel to the y axis and the line passing through the end points, into a parallelogram with two vertical sides and height $(hz)z = hz^2$. A similar statement is true for $\tilde{w}_2$, with the vertical oreintation inverted. It follows that $\tilde{w}_1(R) = w_1(w_1(Q))$, and $\tilde{w}_2(R) = w_1(w_2(Q))$. Then, the desired result follows by induction just iterating on the affine mappings w_1 and w_2.

Remark. The generalization of this lemma to any geometry, with any number of data points and any choice of the parameters d_n is straightforward.

5.2. *Derived measures on dyadic subintervals.*

Lemma 3. Let f be a fractal interpolating function passing by the data points $\{(x_n, y_n): n = 0, 1, 2\}$ with scaling parameters $d_1 = -d_2 = z > 0$. Consider the derived measures $Y(z) = f(U)$ and $Y_j(z) = f_j(U_j)$ with U the uniform measure over $I = [x_0, x_2]$, and U_j the uniform measure over a natural dyadic subinterval I_j, $j = 1, \cdots, 2^{n+1}$ for any n, with f_j being the restriction of f over I_j. Then, the variance of all the $Y_j(z)$ and the variance of Y are approximately equal when $z \approx 1$.

Proof. Let h be any fractal interpolating function with vertical scalings $d_1 = -d_2 = z > 0$ which interpolates three equally spaced data points $\{(\hat{x}_n, \hat{y}_n): n = 0, 1, 2\}$. If $z \approx 1$ it is easily seen from (6) that $h_{2,0}/(\hat{x}_2 - \hat{x}_0) - (h_{1,0}/(\hat{x}_2 - \hat{x}_0))^2 \approx (2\hat{y}_1 - \hat{y}_0 - \hat{y}_2)^2/[48(1 - z^2)]$. The approximation stems from the fact that the numerator shown appears as the limit when $z \to 1$ (e.g. see (13)).

The lemma is proven by realizing that for all dyadic subintervals $|2y_1 - y_0 - y_2| \approx |2\hat{y}_1 - \hat{y}_0 - \hat{y}_2|$, provided that $z \approx 1$. This could be seen by following the images of (x_1, y_1) via w_1 and w_2 (see (9) and (10)). The interpolating function defined in Figure 1 provides a simple example of this fact. In this case, $y_0 = y_2 = 0$, $y_1 = 1$ and then $|2y_1 - y_0 - y_2| = 2$. As explained in Section 2, z^n is added at the nth stage so that if $[\hat{x}_0, \hat{x}_2]$ is a dyadic interval then $\hat{y}_1 = [(\hat{y}_0 + \hat{y}_2)/2] \pm z^n$. Clearly, $|2\hat{y}_1 - \hat{y}_0 - \hat{y}_2| \approx 2$ as $z \approx 1$ for any n.

5.3. *Weightings of Gaussian distributions.* Thus far it has been established that there are derived Gaussians everywhere on the interval $I = [0, 1]$. All derived measures $Y_j(z) = f_{\Theta,D}(U_j)$, based on uniforms U_j defined over dyadic subintervals $I_j \subseteq I$, tend to a Gaussian when $D \to 2$ $(z \to 1)$. Moreover, they all have approximately the same variance when $z \approx 1$, for all n.

Suppose that n is large and let F_X^n be a cumulative distribution which approximates F_X employing $2^{(n+1)} + 1$ dyadic values, $F_X^n(j2^{-(n+1)}) = F_X(j2^{-(n+1)})$, $j = 0, 1, \cdots, 2^{n+1}$. Define $q_j = F_X^n(j2^{-(n+1)}) - F_X^n((j-1)2^{-(n+1)})$ and let U_j be the uniform measure over $I_j = [(j-1)2^{-(n+1)}, j2^{-(n+1)}]$, for $j = 1, \cdots, 2^{n+1}$. Then, Lemmas 2 and 3 give the standardized centered moments of the variables $Y_j(z)$ for $z \approx 1$ (see (14) and (16)),

$$
\begin{aligned}
\mathcal{M}_j(z)^{2m+1} &= \frac{\int_{I_j} f^{2m+1}(x, z)\, dF_{U_j}(x)}{(VY_j(z))^{m+\frac{1}{2}}} \approx \frac{\int_{I_j} f^{2m+1}(x, z)\, dF_{U_j}(x)}{(m_{Y_j}(z)^2)^{m+\frac{1}{2}}} \\
&\approx \frac{2^{n+1} \int_{I_j} f^{2m+1}(x, z)\, dx}{\left(2^{n+1} \int_{I_j} f^2(x, z)\, dx\right)^{m+\frac{1}{2}}} \approx 0,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\mathcal{M}_j(z)^{2m} &= \frac{\int_{I_j} f^{2m}(x, z)\, dF_{U_j}(x)}{\left(\int_{I_j} (f(x, z) - m_{Y_j}(z))^2\, dF_{U_j}(x)\right)^m} \\
&\approx \frac{2^{n+1} \int_{I_j} f^{2m}(x, z)\, dx}{2^{(n+1)m}\left(\int_{I_j} f^2(x, z)\, dx\right)^m} \approx (2m-1)!!
\end{aligned} \tag{19}
$$

The variances $VY_j(z)$ can be safely approximated by the mean squares $m_{Y_j}(z)^2$ since in the vicinity of 1 the variance tends to infinity, while the mean remains constant. For the same reason, the numerators above have been written without subtracting the mean.

The moments of the derived measure $Y(z) = f_{\Theta,D}(X)$ are found weighting those of the variables $Y_j(z)$:

$$
\int_I f^m(x, z)\, dF_X(x) \approx \sum_{j=1}^{2^{n+1}} \int_{I_j} f^m(x, z) 2^{n+1} q_j\, dx
$$

therefore, using (19), the even moments give

$$\mathcal{M}(z)^{2m} \approx \frac{m_Y(z)^{2m}}{VY(z)^m} \approx \frac{\sum_{j=1}^{2^{n+1}} q_j 2^{n+1} \int_{I_j} f^{2m}(x,z)\,dx}{\left(\sum_{j=1}^{2^{n+1}} q_j 2^{n+1} \int_{I_j} f^2(x,z)\,dx\right)^m}$$

$$\approx \frac{\sum_{j=1}^{2^{n+1}} q_j (2m-1)!!\, 2^{(n+1)m} \left(\int_{I_j} f^2(x,z)\,dx\right)^m}{\left(\sum_{j=1}^{2^{n+1}} q_j 2^{n+1} \int_{I_j} f^2(x,z)\,dx\right)^m}$$

$$= \frac{(2m-1)!! \sum_{j=1}^{2^{n+1}} q_j VY_j(z)^m}{\left(\sum_{j=1}^{2^{n+1}} q_j VY_j(z)\right)^m}$$

$$\approx (2m-1)!!$$

since all variances VY_i are approximately equal and since $\sum q_i = 1$. A similar argument, parallel to the one used in Theorem 1, shows that the odd moments vanish when $z \to 1$.

By letting $n \to \infty$ and $z \to 1$, $\int_I f^m(x)\,dF_X^m \to \int_I f^m(x)\,dF_X$ and therefore the theorem follows.

This last theorem may be restated for a family of diffuse probability measures and in particular for the family of binomial multifractal measures.

Corollary 1. Let $X_1, X_2, \cdots, X_n$ be diffuse probability measures with values on the closed interval I. Then, there exists a sequence of fractal interpolating functions $f_{\Theta,D}: I \to R$ such that the derived measures $f_{\Theta,D}(X_1), f_{\Theta,D}(X_2), \cdots, f_{\Theta,D}(X_n)$ all tend to a Guassian when $D \to 2$.

Corollary 2. Let $B(p)$ be a binomial multifractal measure with arbitrary length scales and defined over a closed interval I. Then, there exists a sequence of fractal interpolating functions $f_{\Theta,D}: I \to R$ such that the derived measures $Y_p = f_{\Theta,D}(B(p))$ tend to a Gaussian when $D \to 2$ for all p, $0 < p < 1$.

Remarks. As previously explained, weighing Gaussian distributions should not be confused with adding Gaussian random variables. The remarkable Gaussian property just proven is a consequence of the very peculiar structure of the self-affine fractal interpolating functions of high fractal dimensions. Of course, adding Gaussian random variables gives another Gaussian, but weighing (summing) Gaussian distributions (densities) hardly gives another Gaussian.

Theorem 2 holds for any sign combinations on the scaling parameters d_n and for

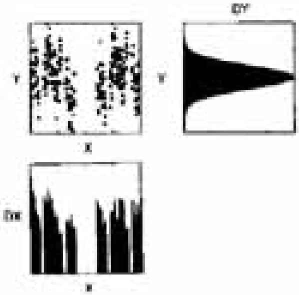

Figure 7. From a Cantorian multifractal measure to the Gaussian via a fractal interpolator. Data points $\{(0,0), (0{\cdot}2, 1), (0{\cdot}4, -1), (0{\cdot}6, -1), (0{\cdot}8, 1), (1,0)\}$; $p_1 = 0{\cdot}3$, $p_2 = 0{\cdot}2$, $p_3 = 0$, $p_4 = 0{\cdot}25$, $p_5 = 0{\cdot}25$, $d_1 = -d_2 = d_3 = d_4 = -d_5 = 0{\cdot}995$.

any number of arbitrarily placed data points. It is easy to generalize all steps of the previous proof if the data points are not equally spaced or if $I \neq [0, 1]$.

Playing the chaos game always defines diffuse measures. If there are more than three data points, then one or more weights p_n could be set to zero yielding measures defined over Cantor sets. Figure 7 illustrates that in fact the Gaussian result is true for Cantorian multifractal measures. All the dots in x vs y align themselves so that they collectively give a Gaussian in y. Of course, the result shown in Figure 7 is valid even if the chaos game is not played to find the figure. In fact, for example, a Cantorian measure is transformed into a Gaussian using a fractal interpolating function passing by three arbitrary data points. And in this case no simple construction via the chaos game could be made.

Figure 5 showed an example of a Gaussian obtained via a binomial multifractal measure with equal length scales and $p = 0{\cdot}3$. If the parent measure is changed but the interpolating function in Figure 5 is kept fixed, different Gaussian measures appear in y. Table 2 shows what the effect is of the multifractal parameter p on the first two moments of the derived measure.

TABLE 2
The mean and standard deviation in y for alternative binomial multifractals. $\{(0, 0), (0{\cdot}5, 1), (1, 0)\}$, $d_1 = -d_2 = 0{\cdot}995$

p	m_Y	σ_Y
0·2	0·200	2·565
0·3	0·302	2·875
0·4	0·399	2·950
0·5	0·500	2·890
0·6	0·599	2·662
0·7	0·694	2·268
0·8	0·791	1·720

6. The Gaussian result and the central limit theorem

It is easy to see that the results proven imply a central limit theorem. As an example, consider three data points $\{(x_n, y_n): n = 0, 1, 2\}$, $d_1 = -d_2 = z > 0$ and define $Y(z) = f_z(U)$, with U being the uniform measure in $[x_0, x_2]$. Now, construct the following random variables:

$$W_1 = Y(0)$$
$$W_2 = Y(3/4) - Y(0)$$
$$\vdots$$
$$W_n = Y(1 - 1/n^2) - Y(1 - 1/(n-1)^2)$$

so that $\sum_{i=1}^{n} W_i = Y(1 - 1/n^2)$. Then, clearly, this sum of random variables (after normalization) converges to a standard Gaussian when $n \to \infty$. This central limit theorem is by no means trivial because it could be easily seen that the random variables W_i are: (a) dependent, (b) non-stationary and (c) unbounded.

It is certainly interesting to know if the result proven here is a simple consequence of a general central limit theorem. If so, such a theorem should accommodate the natural telescopic choice for the summands above. On the one hand, the dependency gives the least of the problems because the W_i being 1-dependent are weakly dependent (see Billingsley (1986)). This allows usage (in principle) of central limit theorems for α-mixing random variables. On the other hand, the other two properties on the W_i do impose severe limitations for the usage of readily available theorems.

Extensive attempts were made to see if the variables W_i satisfied general existing central limit theorems, but no success was attained. To illustrate the difficulties, a general theorem due to Phillip and Stout (1975, pages 95–97) is paraphrased next. It is required first that the W_i satisfy the condition

$$\max_{k \leq n} E\,|W_k - E[W_k]|^{2+\delta} = O(V[W_n]^{\rho\delta/2}) \tag{20}$$

for some $0 < \delta \leqq 2$ and some $0 \leqq \rho \leqq 1/4$, where $E[W_k]$ and $V[W_k]$ are the mean and the variance of the W_k, and O stands for the big o. Also it is required that

$$\sum_{n=M+1}^{M+N} \|W_n - E[W_n]\|_{2+\delta} = O\left(\left\{E\left(\sum_{n=M+1}^{M+N} (W_n - E[W_n])\right)^2\right\}^{\sigma}\right) \tag{21}$$

uniformly in $M = 1, 2, \cdots$ for $\rho\sigma \leqq 1/10$, with $\|\cdot\|_{2+\delta}$ denoting the $2+\delta$ norm $((E[X^{2+\delta}])^{1/(2+\delta)})$.

Due to the nature of the fractal interpolators (see (4)), only the case when $\delta = 2$ could be checked analytically. After some calculations, it is found that the left-hand side in (20) grows much faster than the right-hand side, and consequently the theorem cannot be applied. For other values of δ, a numeric exploration was tried.

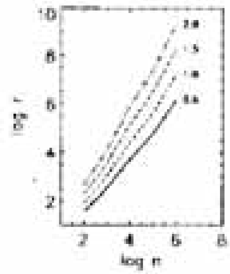

Figure 8. Ratio of the left-hand side over the right-hand side of (20) as a function of δ. $\rho = 0{\cdot}25$.

All cases considered also gave much faster growth on the left-hand side of (20), leading again to the conclusion that such a general central limit theorem cannot be applied to prove our result. Figure 8 shows some examples of the ratio of the left-hand side over the right-hand side, r, as a function of n. As is seen, there is a power law increase in all cases with larger ratios for higher δ.

It is not known at this time if there is a generalized central limit theorem (other than that of Phillip and Stout (1975)) which covers the limiting Gaussian results of this work. This is an open question. It could be that our result, proven via the moments, may lead to generalized central limit theorems as the one in this section, or it could indeed by that it fits a very general central limit theorem.

7. Final remarks

It is always possible to transform any diffuse probability measure into a Gaussian by means of a transformation T. What is truly remarkable is that the fractal functions $f_{\Theta,D}$ described in this work lead to Gaussians when $D \to 2$, regardless of the initial diffuse probability measure. The fractal interpolating functions of high fractal dimensions are therefore intrinsically Gaussian. As far as we know, these fractals are the first known non-random mathematical objects with such a property. Generalizations to higher dimensions may also be carried out so that bivariate Gaussian measures are obtained; see Puente and Klebanoff (1994).

Although other 'non-random' constructions of the Gaussian distribution exist, e.g. via dynamical systems with enough mixing (see Beck and Roepstorff (1987) and Ratner (1973)), the one provided here is remarkably simple. The construction is particularly appealing because it is based on geometric objects which have increasing relevance in the study of nature (i.e. fractals and multifractals). The connection between 'disorder' and 'harmony', as exemplified in Figure 5, further stresses the relevance of the new construction.

As the magnitude of the scaling parameters d_n is increased, and as alternative parent binomial (multinomial) multifractals are induced, a wide variety of derived measures appear in y (see Figure 3 for the uniform case). These derived measures include multifractal measures (other than the natural multinomial multifractals) and for high enough fractal dimensions measures which appear to be absolutely continuous. Given their connections to fractals and turbulence, such derived distributions represent viable alternatives for modeling nature's processes. This idea is supported by the fact that the two extreme cases (Gaussian and multifractals) could be constructed within this framework.

Appendix A. Calculations of inductive proof for even moments

Equation (17) may be written expanding the coefficients $(s_{i,j})$ in terms of the parameters of the affine mappings as follows:

$$\begin{aligned}\frac{f_{2m+2,0}(z)}{VY(z)^{m+1}} \approx \{&(2m+1)(m+1)(a_1 f_1^2 + a_2 f_2^2) z^{2m} f_{2m,0}(z)\\ &+ (2m+1)(2m+2)(a_1 c_1 f_1 + a_2 c_2 f_2) z^{2m} f_{2m,1}(z)\\ &+ (2m+1)(m+1)(a_1 c_1^2 + a_2 c_2^2) z^{2m} f_{2m,2}(z)\\ &+ (2m+2)(a_1 f_1 - a_2 f_2) z^{2m+1} f_{2m+1,0}(z)\\ &+ (2m+2)(a_1 c_1 - a_2 c_2) z^{2m+1} f_{2m+1,1}(z)\} \frac{1}{VY(z)^{m+1}(1-z^{2m+2})}.\end{aligned} \tag{A.1}$$

Each of the last four rational functions may be written in terms of $f_{2m,0}(z)$ for z in the vicinity of 1. The following expressions are found using Lemma 1.

$$f_{2m,1}(z) \approx \frac{(a_1 e_1 + a_2 e_2) z^{2m} f_{2m,0}(z)}{1 - (a_1^2 + a_2^2) z^{2m}} \tag{A.2}$$

$$f_{2m,2}(z) \approx \frac{\{(a_1 e_1^2 + a_2 e_2^2) z^{2m} f_{2m,0}(z) + 2(a_1^2 e_1 + a_2^2 e_2) z^{2m} f_{2m,1}(z)\}}{1 - (a_1^3 + a_2^3) z^{2m}} \tag{A.3}$$

$$f_{2m+1,0}(z) \approx (2m+1)(a_1 f_1 + a_2 f_2) z^{2m} f_{2m,0}(z) + (2m+1)(a_1 c_1 + a_2 c_2) z^{2m} f_{2m,1}(z) \tag{A.4}$$

$$\begin{aligned} f_{2m+1,1}(z) \approx\ &(2m+1)(a_1 e_1 f_1 + a_2 e_2 f_2) z^{2m} f_{2m,0}(z)\\ &+ (2m+1)(a_1^2 f_1 + a_1 c_1 e_1 + a_2^2 f_2 + a_2 c_2 e_2) z^{2m} f_{2m,1}(z)\\ &+ (a_1 e_1 - a_2 e_2) z^{2m+1} f_{2m+1,0}(z) + (2m+1)(a_1^2 c_1 + a_2^2 c_2) z^{2m} f_{2m,2}(z).\end{aligned} \tag{A.5}$$

Replacing all these equations back into (A.1), and using the values for the affine mapping parameters as implied by (9) and (10) one gets,

$$
\begin{aligned}
\frac{f_{2m+2,0}(z)}{VY(z)^{m+1}} = \Bigg\{ & \frac{y_1^2 z^{2m}}{2} + \frac{(y_2 - y_1 + zy_2)y_1 z^{4m}}{(4-2z^{2m})} + \frac{(zy_2 - y_1)^2 z^{2m}\left(\frac{z^{2m}}{8} + \frac{z^{4m}}{16-8z^{2m}}\right)}{\left(2 - \frac{z^{2m}}{2}\right)} \\
& + \frac{(y_2 - y_1 + zy_2)^2 z^{2m}\left(\frac{z^{2m}}{8} + \frac{z^{4m}}{16-8z^{2m}}\right)}{\left(2 - \frac{z^{2m}}{2}\right)} - y_1 z^{2m+1}\left(\frac{y_1 z^{2m}}{2} + \frac{y_2 z^{4m}}{8-4z^{2m}}\right) \\
& - (2zy_2 - 2y_1 + y_2)z^{4m+1}\left(\frac{y_1}{4} + \frac{y_2 + zy_2}{16-8z^{2m}}\right) \\
& + \frac{(2zy_2 - 2y_1 + y_2)z^{4m+2}}{4}\left(\frac{y_1 z^{2m}}{2} + \frac{y_2 z^{4m}}{8-4z^{2m}}\right) \\
& - \frac{y_2(2zy_2 - 2y_1 + y_2)z^{4m+1}\left(\frac{z^{2m}}{8} + \frac{z^{4m}}{16-8z^{2m}}\right)}{(4 - z^{2m})} \Bigg\} \frac{(m+1)(2m+1)f_{2m,0}(z)}{VY(z)^{m+1}(1-z^{2m+2})}.
\end{aligned}
\tag{A.6}
$$

A recursive equation of $\mathcal{M}(z)^{2(m+1)}$ as a function of $\mathcal{M}(z)^{2m}$ is obtained replacing one term $VY(z)$ on the right-hand side by $t(z)/48(1-z^2)$ according to (13). The limit when $z \to 1$ is obtained term-by-term, expanding the quotient $1/(1-z^{2m+2})$ as $1/((1-z^2)(1+z^2+\cdots+z^{2m}))$, and noticing that

$$
\lim_{z\to 1} \frac{1}{VY(z)(1-z^{2m+2})} = \frac{48}{(m+1)(2y_1 - y_2)^2}.
$$

The limit is

$$
\begin{aligned}
\lim_{z\to 1} \mathcal{M}(z)^{2m+2} = & \{[24y_1^2] + [24y_1(2y_2 - y_1)] + [8(y_2 - y_1)^2] + [8(2y_2 - y_1)^2] \\
& - [12y_1(2y_1 + y_2)] - [12(3y_2 - 2y_1)(y_1 + y_2)] \\
& + [3(3y_2 - 2y_1)(2y_1 + y_2)] - [4y_2 3y_2 - 2y_1)]\} \frac{(2m+1)}{(2y_1 - y_2)^2} \lim_{z\to 1} \mathcal{M}(z)^{2m} \\
= & (2m+1) \lim_{z\to 1} \mathcal{M}(z)^{2m}.
\end{aligned}
\tag{A.7}
$$

Consequently, if $\lim_{z\to 1} \mathcal{M}(z)^{2m} = (2m-1)!!$ then $\lim_{z\to 1} \mathcal{M}(z)^{2m+2} = (2m+1)!!$

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