THE HYPOTENUSE: THE PATHWAY OF PEACE

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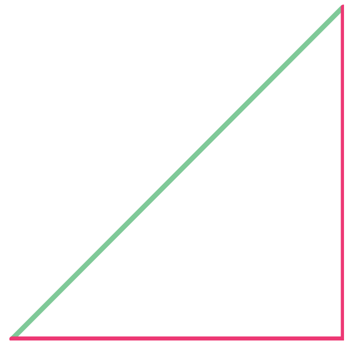
Outline

- *Reviews the famous Pythagorean theorem for right-angled triangles.*
- *Introduces simple multiplicative cascades that break equilibrium into fractal dust.*
- *Introduces the so-called “devil’s staircase” and its universal properties.*
- *Explains how fully developed turbulence happens in the air via a generic cascade.*
- *Illustrates that such a process is always short-lived as it is dissipative.*
- *Shows how generic cascades may be used to illustrate systems of the world.*
- *Argues that it is best to avoid the “turbulence” of any divisive cascade in our lives.*
- *Shows that peace is found in serenity and love and symbolized by the hypotenuse.*

- Shown below is an isosceles *right-angled triangle*, whose *legs* measure one unit each:



- In virtue of the Pythagorean theorem, the square of the *hypotenuse* equals the sum of the squares of the legs, that is, two.

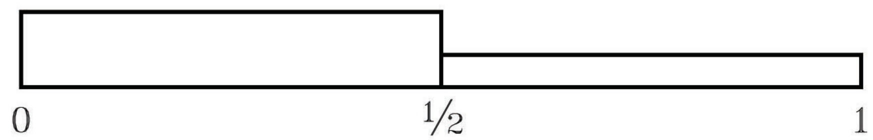
- As such, the shortest distance between the *top* and *bottom* of the triangle, gives a value of $\sqrt{2} = 1.4142\cdots$.

- As we shall see and despite our inability to fully understand "dot, dot, dot" on such an irrational number, the *root of two* is a rather powerful and symbolic concept.

- To show that such is the case and that the 45 degree *straight line* is the *pathway of peace*, it is convenient to introduce first a simple game that kids learn to play molding modeling clay, as follows.

- Start with a *uniform* bar of clay, as it comes out of the box, and cut it by its $p = 70\%$:

- Then, mold the two pieces, *piling up* the one on the left and *stretching* the other on the right, both towards the *middle*, so that they make two contiguous bars of equal horizontal length:

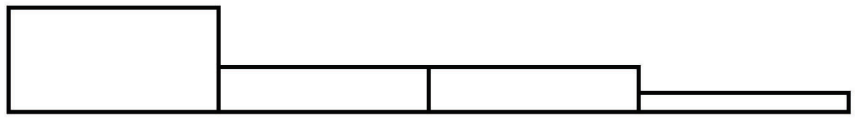


- The height of the piece on the left is higher than the original, and the hight of the one to the right is smaller.

- If the original bar has a height of *one* unit, the "rectangle" on the left has $2{\cdot}p = 1.4$ *vertical units*, for its area, i.e., the mass of the piece, computed multiplying its base $1/2$ times its height, is $p = 70\%$ of the original.

- Similarly, the height of the rectangle on the right is $2 \cdot q = 0.6$ vertical units.

- With the process fully understood, repeat it on each bar, with exactly the same proportions, to obtain four pieces of horizontal size $1/4$:



- The masses (areas) of such pieces are respectively, 70% of 70%, 30% of 70%, 70% of 30%, and 30% of 30%, which gives, *multiplying*, 49, 21, 21, and 9% of the total mass.

- Such correspond to the familiar expansion of $(p+q)^2$: p^2, twice $p \cdot q$ and q^2.

- Now the tallest rectangle has a height of $(2 \cdot p)^2 = 1.4^2 = 1.96$ vertical units.

- Clearly, as the process is repeated for a total of n levels, the number of pieces increases in *powers of two*: one gets 2^n rectangles all having horizontal sizes $1/2^n$, whose masses, adding always 100%, turn out to correspond to the expansion of $(p+q)^n$. (!)

- The process, known as a **multiplicative cascade**, defines $(n+1)$ heights of rectangles arranged by *layers*: p^n, $p^{n-1} \cdot q$, $\cdot \cdot \cdot$, $p \cdot q^{n-1}$, and q^n. There is one piece having p^n of the mass and one having q^n, there are n rectangles with $p^{n-1} {\cdot} q$ and $p {\cdot} q^{n-1}$, $n {\cdot} (n-1)/2$ pieces with $p^{n-2} \cdot q^2$ and $p^2 \cdot q^{n-2}$, and so on, according to the well known *Pascal's triangle*.

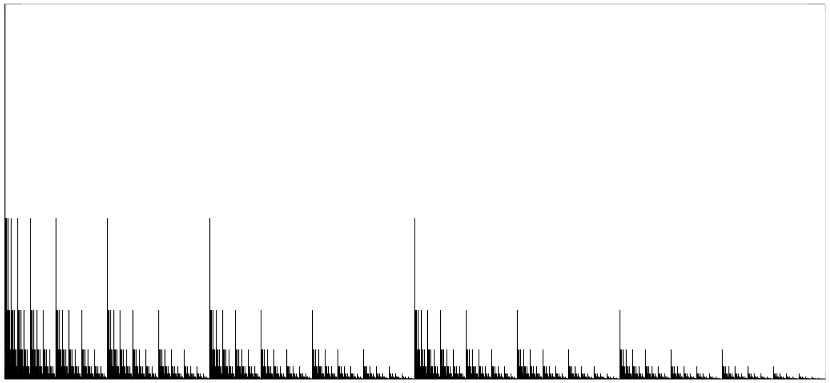
1
1 1
1 2 1
1 3 3 1

- The number of rectangles on layers $p^k \cdot q^{n-k}$ and $p^{n-k} \cdot q^k$ are given by *binomial coefficients*:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!\,(n-k)!},$$

where $k! = k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 2 \cdot 1$ denotes the *factorial* operation, and $0! = 1$.

- When $n = 12$, the following set of 4,096 rather thin and hence *spiky* rectangles is found:

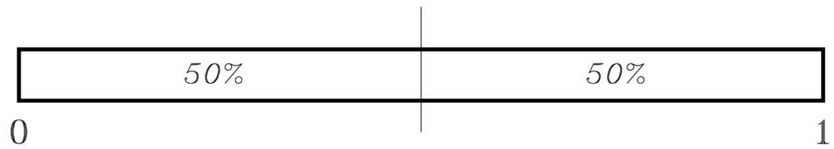


- This set is not drawn at scale, for its vertical scale is $1.4^{12} = 56.69$ vertical units. (!)

- As may be appreciated, the *divisive* and rather *simple* game for kids destroys **equilibrium** into dispersed *thorns* exhibiting a remarkably *intertwined structure.* (!)

- Although the mass is *conserved*, moving from *thorn* to *thorn* is rather difficult, for, as the process continues, rectangles of equal size are *seldom contiguous*.

- To further appreciate the *empty* structure generated within each layer by this cascade, it is convenient to introduce yet another game for kids.
- This one starts as before, with a *uniform* bar of modeling clay, but this time one cuts it by the *middle*:



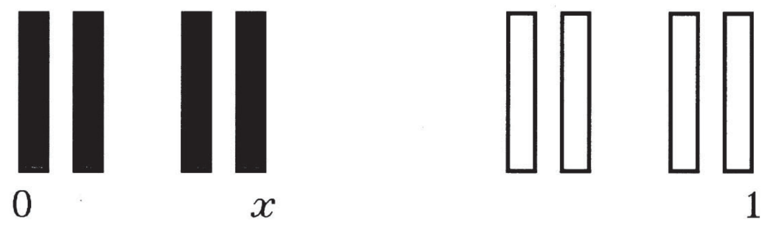
- Now, the two pieces are separated and *piled up* left and right, so that their horizontal lengths equal $1/3$ and such that each piece contains 50% of the original mass:

- The rectangles, having a *gap* between them, have a common height of 1.5 vertical units.
- As before, the game continues on repeating the process on each piece, cutting and separating according to the same proportions.
- This clearly yields another divisive *multiplicative cascade* that produces, after n levels of the construction, 2^n equal and dispersed rectangles having equal masses of $1/2^n$, horizontal lengths $1/3^n$, and *diverging* heights $(3/2)^n$.

- As the process is repeated, the game generates *spikes* that do not touch, *thorns* that grow to *infinity* and which are supported by *Cantor dust*. (!)

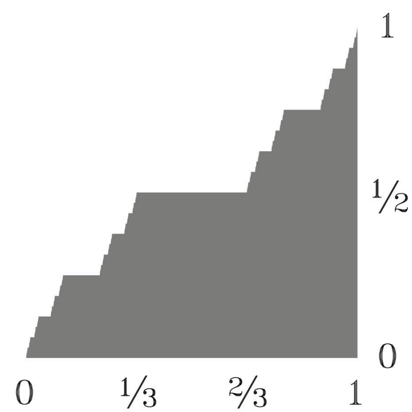
- In the limit, all the infinitely many spikes contain *nothing*, but altogether the masses add up to 100%. Remarkably in this case, and also for the first cascade, $\infty \cdot 0 = 1$. (!)
- By varying the *size of the hole* on the second game, one may capture the *non-contiguous*, *empty* and *fractal* structure present on all *layers* generated by the first cascade.
- While less dense layers, towards the periphery of Pascal's triangle, correspond to larger gaps, the more dense layers require smaller and smaller holes.
- Due to the presence of a multitude of *dusts* (one per *layer*) which decompose the domain of the spiky object of the first game, such a set is known in the literature as a **multi-fractal**.
- While the second cascade divides equilibrium into an infinite set of equal spikes supported by *dust*, the first one gives intertwined *"multi-thorns"* over *"multi-dust."* As such, the two divisive games are inherently related to one another.

- To further appreciate the objects generated by both games and as such grow without bound, it is convenient to portray their *accumulated masses*, from the beginning up to a point x:

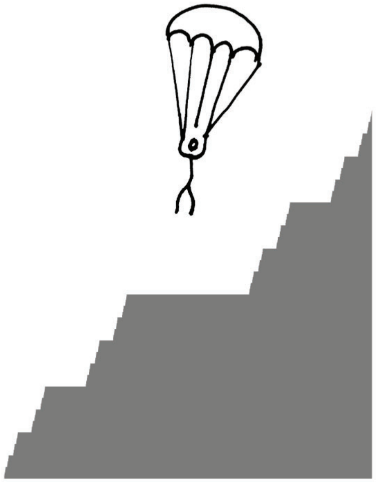


- For the cascade with gaps, the *wealth* up to point x, $W(x)$, gives: $W(0) = 0$, $W(1) = 1$, $W(1/3) = 1/2$, $W(2/3) = 1/2$, and also for any value of x in the main gap, $W(1/9) = 1/4$, $W(2/9) = 1/4$, and so on.

- At the end, $W(x)$ vs. x contains *plateaus* wherever the original object had *gaps*:



- It happens that such an object is quite peculiar, for if we were to parachute on it from the top, it would appear to us, once we arrive, that it is *flat*.

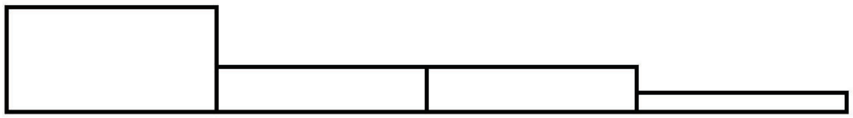


- If our "scale" is small enough, we would touch down on a plateau with all likelihood and we would falsely believe to have landed in *equilibrium*.

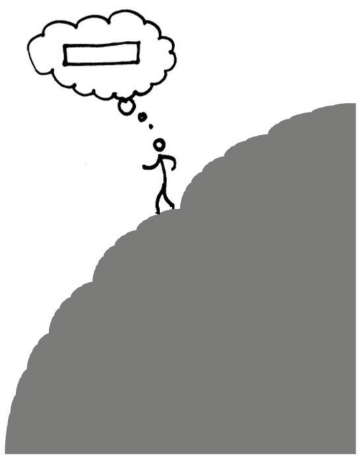
- Due to such a *deception* and due to the *divisive* nature of the generating cascade, such a *continuous* object having *no derivatives* at great many places was properly named by George Cantor in 1883 as the "**devil's staircase**." (!)

- As the stair contains only horizontal or vertical steps (seeing any inclined lines above is just an illusion), the length from top to bottom, following such a *jagged* curve, equals **2 units**: one horizontal unit for all gaps plus one vertical unit, as the cascade conserves mass.

- This property is **universal**. The second cascade, irrespective of the initial size of the *hole*, h, yields a wealth function that is a *devil's staircase* with *maximal* length of 2 units. (!)
- For the first cascade the accumulated wealth may be calculated easily following the dynamics of the divisive game.

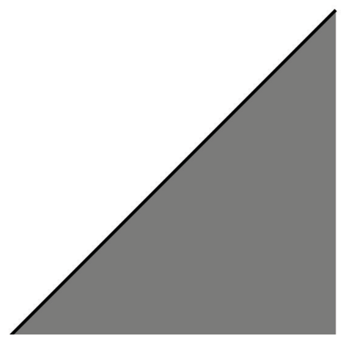


- For instance, after two levels: $W(1/4) = 0.49$, $W(1/2) = 0.7$, $W(3/4) = 0.91$, and so on.
- This yields another *devil's staircase* shaped as a *cloud of dust*:



that is, another jagged boundary of *maximal length*, for it is locally flat everywhere. (!)

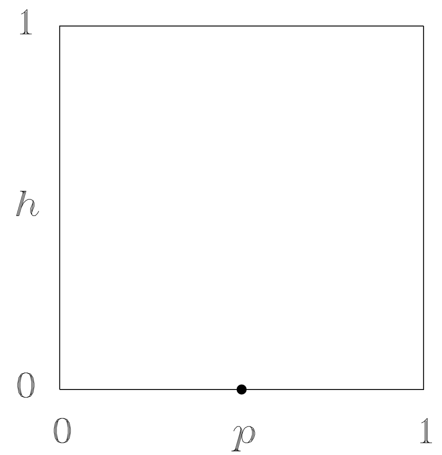
- This property is also **universal** for the first cascade, for such always results in a *continuous* boundary having horizontal-vertical notches and hence *no derivatives everywhere*. If there is any *imbalance* $p \neq 1/2$, no matter how small, such a game grows *thorns* separated in *layers* over *dust*, and such give wealth stairs whose lengths are always **2 units**. (!)
- There is a escape however. When the first game is played by the *middle*, i.e., at $p = 1/2$, and the second one is performed *without holes* at $h = 0$, then the *balance* of the original bar is always maintained: 
- In such a case, the wealth frontier simply gives a **straight** ramp having **minimal** length from top to bottom:



- While the keeping of **equilibrium** yields a wealth function that travels through **the hypotenuse**, the divisive games give jagged stairs that are as long as the **legs** of an isosceles right-angled triangle. (!)

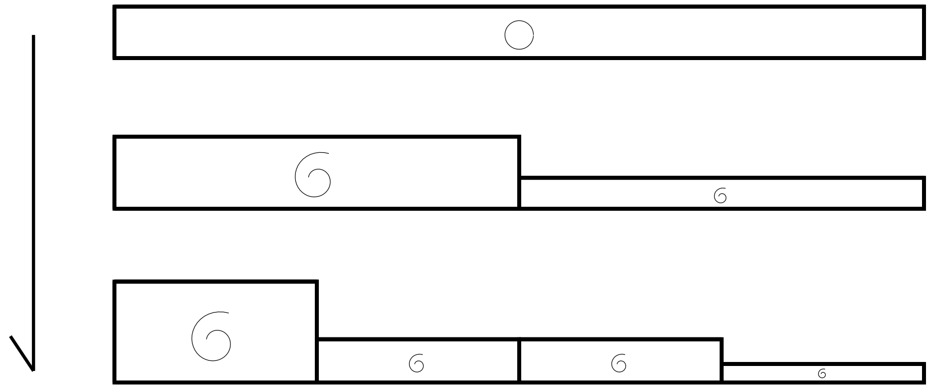
- When the two divisive games are combined to produce additional cascades containing both *imbalances* p and *holes* h, other more exotic sets of *thorns* over *dust* and subsequent *devil's staircases* are produced.

- As shown below, there is only **one point**, within a square of possibilities, that leads to the minimal distance of $\sqrt{2}$:



- But the matter is yet more dramatic, for the bar may be split in more than two pieces and the cascades may be played aided by *chance*, i.e., using variable imbalances and holes in the process, and such general mechanisms would also result in *thorns* over *dust* and in *devil's staircases*. (!)

- It happens that recent technological advances have allowed identifying the progressive breaking given by the first cascade in the way **turbulence** happens in the *air*.



- When the air's *inertia* exceeds the internal *cohesion* of the fluid, i.e., when its *Reynolds number* $\mathcal{R} = v \cdot L/\nu$ is large, the air flows in an *irregular* and *intermittent* fashion.

- As it happens to other fluids, when the air's *viscosity* ν is overpowered by excessive energy, as measured by the product of the flow's velocity v and a characteristic length L, the fluid can no longer flow as *a whole* and in a *laminar* and *calm* way, but rather splits into *inwardly rotating* elements called *eddies*.

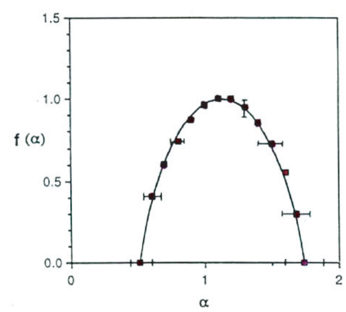
- A *cascade*, as first introduced by Lewis Fry Richardson in 1922, forms and the *spiraling* sets carry distinct amounts of *kinetic energy* from place to place.

- As the process continues, the energies arrange into *intermittent* and **violent** outbursts that eventually **dissipate** in the form of *heat*, when the eddies reach a small enough scale.

- What is observed in experiments along one dimension are *intertwined layers of energies* that are **universally** consistent with those produced by the first *multiplicative cascade*, when the offspring eddies carry *precisely* 70 and 30% of the parent's energy. (!)

- Remarkably and as first reported by Charles Meneveau and Katepalli Sreenivasan in 1987, nature yields a **permutation** of the generic cascade, with the more energetic eddies happening not always to the left but either to the left or to the right, as guided by chance.

- For grid turbulence, wake of a cylinder turbulence, boundary layer turbulence, and atmospheric turbulence one gets:



a truly excellent fit, with the squares and bars associated with mean and deviations of data and the inverted parabola corresponding to the simple cascade with $p = 0.7$. (!)

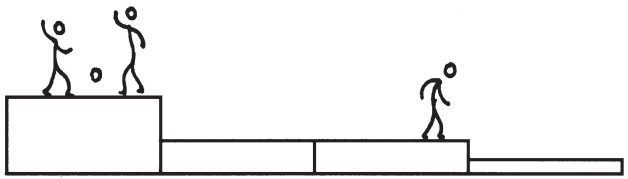
- One day, while pondering these remarkably simple and far reaching results, valid for eddies from Afghanistan to Zimbabwe, it occurred to me that the cascade notions could also be used to describe, at least figuratively, the ways we humans create our own "*turbulence*."

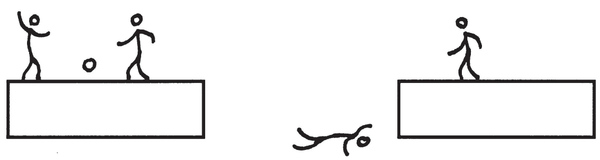
- After all, we are all faced with "*inertial forces*" that often break our "*cohesions*" and, when that happens, such also leads to "*intermittent*" behavior and to the pain of **violence**.

- As our distress (and certainly mine) many times is associated with the relentless *repetition* of a *divisive* trait, it also seemed reasonable to me to employ the two general cascades to symbolize the pathways that lead us (me) to "*bite the dust.*" (!)

- Given the *universality* of the notions and the clear "*self-similarity*" of our division, I believe such simple geometric ideas are useful to describe our "*distress*" at a variety of scales: within ourselves, in our relationships, in our societies, and in the world at large.

- The two cascades are, I sense, accurate metaphors to contemplate the most common ways we employ to propagate division: **inequities** and **discriminations**: (!)

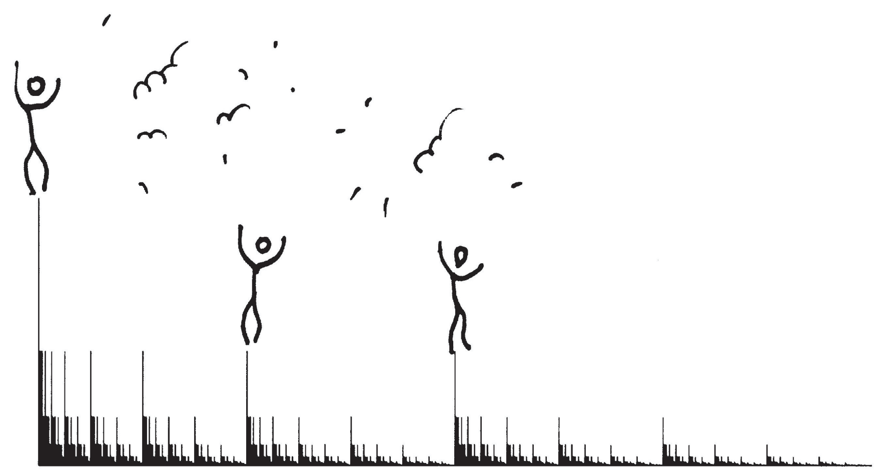




- And such notions may be used, safely I believe, to vividly expose the inherent fallacies in the systems that have governed our humanity.

- For it is clear from recent history, and certainly from common sense, that "*equality by force*" does not work, for the implicit *fear* "in ending up in a gap" **dissipates** the fabric of *friendship* and *love* that may truly sustain a society.

- And for although the system based on the notion of the "*survival of the fittest*" appears to have won the battle,



such also produces *thorns* and *dust* and also destroys *friendship*, in its implacable search "*to be number one*."

- The ideas turn out to have an unexpected validity beyond the metaphorical, for the *multi-fractal* object generated by the first cascade, for $n = 20$ and $p = 0.7$ (as in nature), closely matches the distribution of wealth on the most powerful nation on earth. (!)

- In fact, simple calculations via Pascal's triangle reveal a close fit of the wealth of the richest 5, 10, 20, and 40% in the **United States** as reported in 1998, that is, in order, 59 (57), 71 (70), 84 (84), and 95% (95) of the resources, with the cascade values given in parenthesis.

- Although the wealth of the richest 1% is underestimated by the simple cascade, 38% (30), these results allow us to visualize the dreadful consequences of propagating a **cascade** and provide a poignant call to **change** in order to avoid the prescribed *dissipation* implied by the physics of nature and common sense. (!)

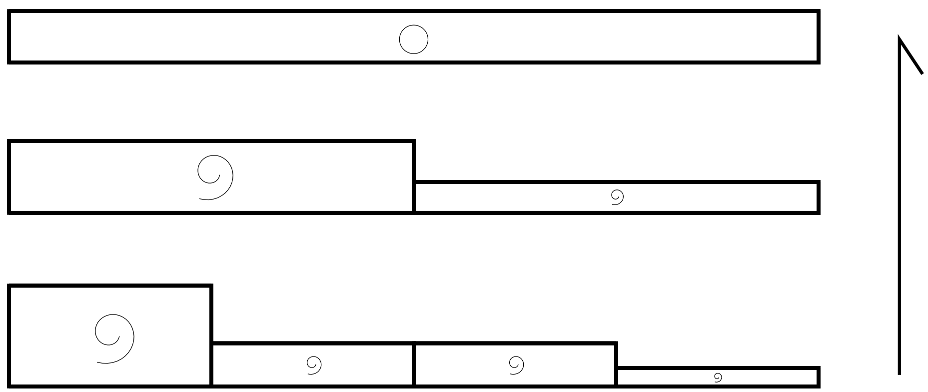
- For the Christian notion of "loving one another" contradicts the reality of "competing against one another" and for the latter brings *misery* and *loneliness*, even to those "at the top."

- As the cascade notions are certainly *universal*, with them one may fit (even if by adding chance to the process) the "*heavy-tailed*" distributions of wealth of any nation, and also the overall distribution of wealth of our world, one that sadly includes 2/3 of its population living under extreme poverty. (!)

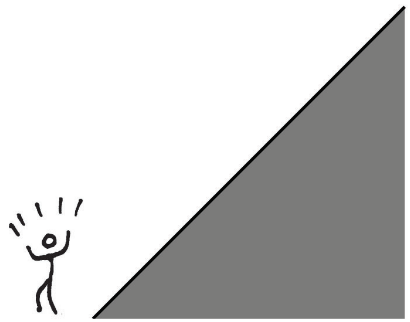
- With such, one may readily recognize a common tread for such *power-laws*, also known as *Pareto distributions* after Vilfredo Pareto who first noticed their ubiquity in 1906: all such distributions may be thought of as *devil's staircases* traveling maximal lengths. (!)

- For even if statistical qualifiers, which measure the well-being of human beings by finite numbers, tell us that some divisive distributions are better than others and that *globalization* may bring *peace* and *justice* to all, the *cascade* notions remind us of our historical *evil* and *greed* and point us to **the point** of *true equilibrium* where we may indeed fulfill our very essence. (!)

- For division can be defeated, as we, humans with a *soul*, may learn from nature in order to run a cascade in reverse to heal our "fractal" world:



- For, we may faithfully "parachute" on the *unitive* **hypotenuse**, $Y = X$, and use its *simple* and *radical wisdom* to avoid all *thorns* and *dust*.



- For figuratively speaking but also with precise certainty, there is no other way for us to solve the problems we face but to arrive at the "**origin**" of things. (!)
- At the end, I believe that much can be learned from the simple cascades, relevant lessons that surely apply to each one of us.
- To me, the many *universal symbols* in this lecture remind us of our inherent **choices**:
- We may select *equilibrium* or *turbulence*, *calmness* or *violence*, *rectitude* and **peace** or the "*wickedness*" that comes when we decide, even if inadvertedly, to live our lives "stressed" and at high Reynolds numbers. (!)

- The images herein clearly reaffirm the goodness of the proverbial "*fifty-fifty*" as essential for *friendship*, for its *shortest* and *straight* path is far better than any form of *inequities* and *holes* that take us to *longest* separations. (!)

- For married couples do find *happiness* when they dare to encounter their common *root*. (!)

- These simple ideas also remind us that we may choose to reverse the **natural** process to be *reconciled* with one another, i.e., *integration* and its beautiful symbol $\int$ (the slender "S"), instead of continuing believing in *separation* and *division* and its symbol \$, a clear "*negative log*" that prevents us from recognizing each other as brothers and sisters. (!)
- For we may indeed select *wholeness* over *emptiness*, *unity* over *brokenness* (*fractal dust*), and, quite vividly, an infinite sequence of *outwardly* rotating elements denoting our **unity** and **Love**, $1 = 0.999\cdots$, over an "**evil fraction**," $2/3 = 0.666\cdots$, that may be read from the divisive natural cascade and that properly symbolizes the *foolishness* and *emptiness* of, what is defined in the dictionary, as our **sinfulness**. (!)

- For as may be seen on a clock, the ever *positive* spiral of unity, $r = e^{+\theta}$, always looks ahead to the *future* with mercy, while division is based on a *negative* spiral, $r = e^{-\theta}$, one rooted in the *past* that, if not tamed, always looks for revenge. (!)

- To conclude, these are some of the options that I believe may be drawn from this lecture:

Equilibrium	*Turbulence*
Calmness	*Violence*
Rectitude	*Wickedness*
"fifty-fifty"	*Inequity*
Shortest	*Longest*
Reconciliation	*Separation*
Integration, $\int$	*Division,* $
Wholeness	*Emptiness*
Unity	*Dust*
$1 = 0.999\cdots$	$2/3 = 0.666\cdots$
Positive, +	*Negative, -*
Future	*Past*

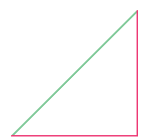
- The following poem-song summarizes this lesson.

THE HYPOTENUSE

By the wisdom of science
simply divides the air,
to dissipate all its heat
coding a subtle cascade.

Turbulence is selfish game
for it scatters the whole,
and its sequence is a frame
for the options of the soul.

Two options before us
two pathways ahead,
the one is the longest
the other straight.



We journey directly
or go by the legs,
we follow intently
or end up in pain.

By walking the flatness
or hiking the spikes,
we travel in lightness
or take serious frights.

The incentive is unity
and the call proportion,
the key is forgiveness
and the goal true notion.

In wandering wickedness
there is never a fruit,
but in ample humbleness
one encounters the root. $\sqrt{2}$

There is no excuse,
let's practice fair game:
it's by the hypotenuse
or else by the legs.

There is no solution
but walking straight:
the spikes of disorder
insinuate the way. (2)

There is a best pathway:
the palpably smooth.

It's by the hypotenuse
and walking in truth.

There is one solution,
I tell you the truth. $Y = X$

It's by the hypotenuse
and walking in truth.

For any other pathway
will lead us astray.

It's by the hypotenuse,
there is no other way.

Oh listen, you brother,
let's brighten the day.

It's by the hypotenuse,
there is no other way.

$2/3 = 0.666\ldots$

Otherwise, the devil
shall pull by the legs.

It's by the hypotenuse
or else by the legs.

For such road is fractal:
as long as it gets.

It's by the hypotenuse
or else by the legs.

Oh let's mend the broken,
growing to the root.

It's by the hypotenuse,
the one that yields fruit.

$1 = 0.999\ldots$

Let's keep equilibrium,
avoiding dark soot.

It's by the hypotenuse,
the one that yields fruit.

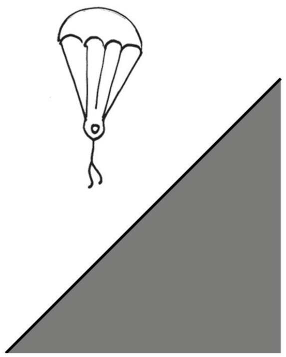
Oh listen, you brother,
a counsel from science.

It's by the hypotenuse:
the simplest design.

I tell you integrating,
don't leave it to chance. $\int$

It's by the hypotenuse:
the simplest design.

It's by the hypotenuse:
the simplest design. (2)



References:

1. J. Feder, *Fractals*, Plenum Press, New York, 1988.
2. B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, New York, 1982.
3. C. Meneveau and K. R. Sreenivasan, "Simple multifractal cascade model for fully developed turbulence," *Phys. Rev. Lett.* 59:1424, 1987.
4. C. E. Puente, *The Hypotenuse. An illustrated scientific parable for turbulent times*, 2005.
4. K. R. Sreenivasan, "Fractals and multifractals in fluid turbulence," *Ann. Rev. Fluid Mech.* 23:539, 1991.