Chaos, Complexity & Christianity

5. The deterministic nature of chaos

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Summary

- *Introduces the logistic map and its amazing dynamics.*
- *Explains how such a deterministic equation gives rise to intertwined periodic and chaotic behaviors.*
- *Introduces the diagram of bifurcations or the Feigenbaum tree.*
- *Explains why the "butterfly effect" happens.*
- *Shows chaotic attractors in two and three dimensions.*

The dynamics of the logistic map

(May, 1976; Gleick, 1987; Schroeder, 1992; Turcotte, 1997)

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$$X_{k+1} = \alpha X_k (1 - X_k)$$

where X is the normalized size of a population (between 0 and 1), say of ***rabbits***, k and k+1 are two successive generations and α is a ***parameter*** that may be between 0 and 4, inclusive.

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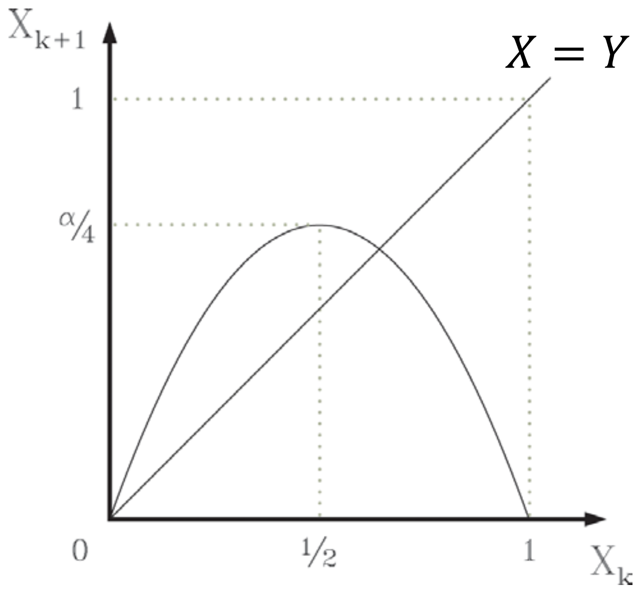
• The quadratic equation defines, from a generation to the next, a symmetric graph with the form of a ***parabola***, one that passes by the points (0,0) and (1,0) and whose peak, by the middle, is $\alpha/4$:

The logistic map



• The curve exhibits and increase from generation to generation if the population is small, but a decrease if the population is large, which is ***logical***.

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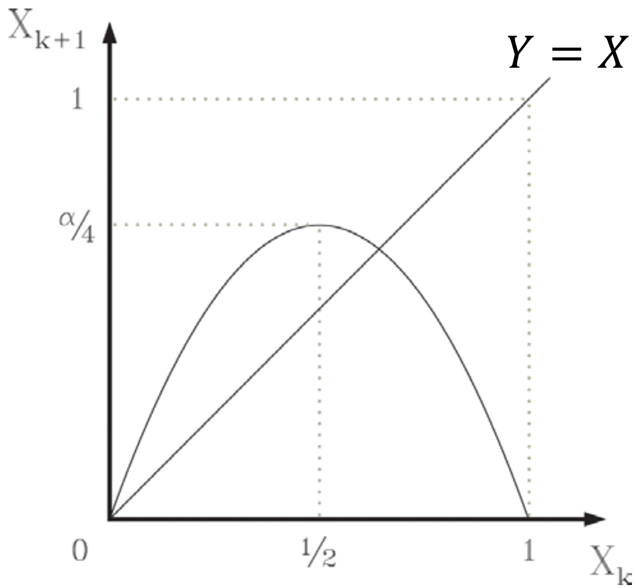

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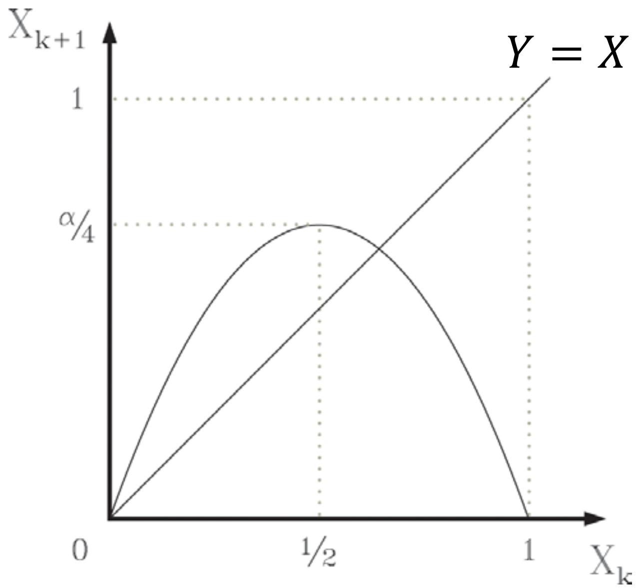
- The straight line ***Y = X*** has been added to the figure to calculate the evolution of a population that starts at a size X_0: the next size is read from the graph, and then such X_1 is taken to the one-to-one line to read X_2, etc.

The logistic map



$\alpha = 2.8$

- Here we observe the evolution of a population, ***reiterating*** the logistic map, when the parameter takes on the value of 2.8.

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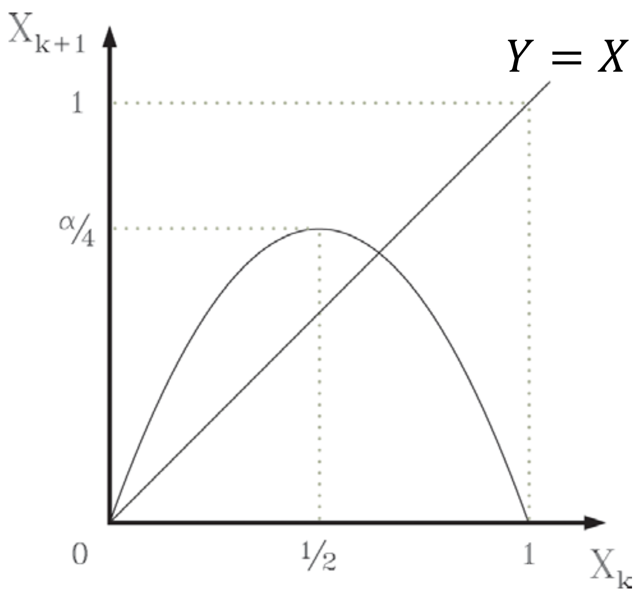

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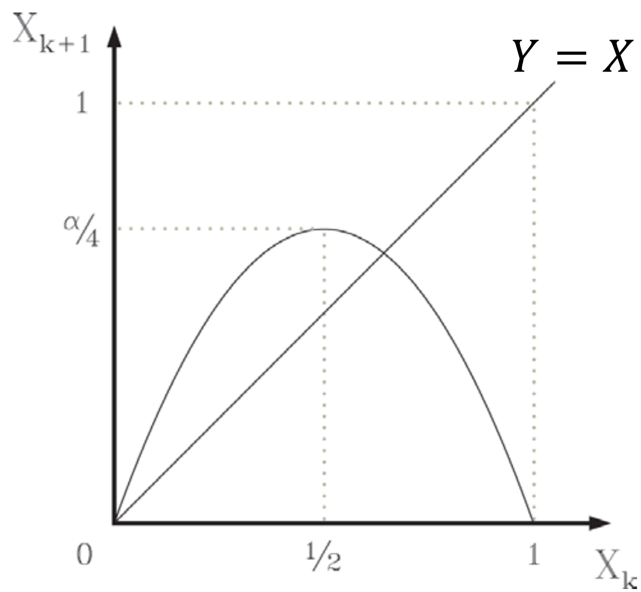
- As may be seen, the population converges to a value X_∞ that is the non-zero intersection between the straight line and the parabola, and this "***attractor***" always happens provided that X_0 is not 0 or 1.

The logistic map



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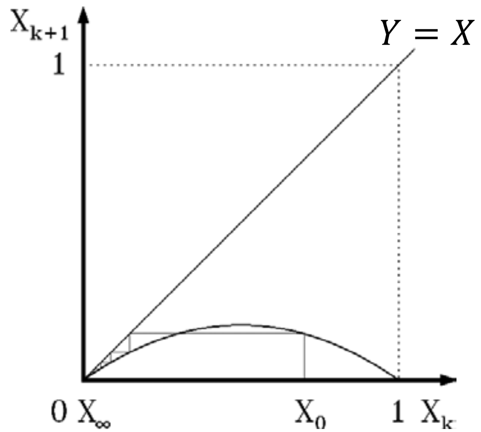
- But this is not always the case, and what is obtained depends on α:

The logistic dynamics

$0 < \alpha \leq 1$

$X_\infty = 0$



• When the parabola is below the line, that is when α is less than or equal to 1, the population becomes ***extinct*** and the ***origin*** attracts the dynamics for every initial size X_0.

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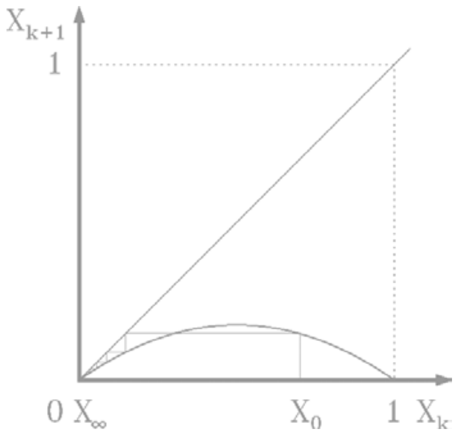

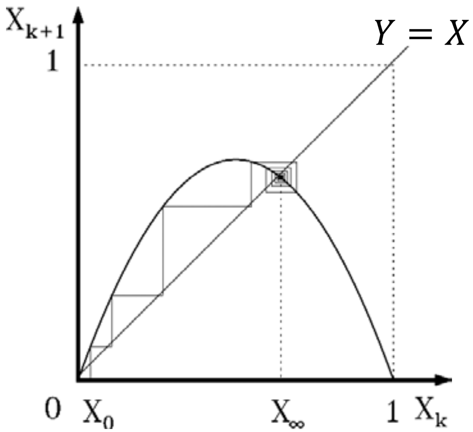

$1 < \alpha \leq 3$

$X_\infty = \frac{\alpha - 1}{\alpha}$

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- Now, when the curve "***crosses the line***" and α is between 1 and 3, the population converges to the non-zero intersection between the line and the parabola, that is, to the "***fixed point***" given by the shown equation, alpha minus one over alpha.

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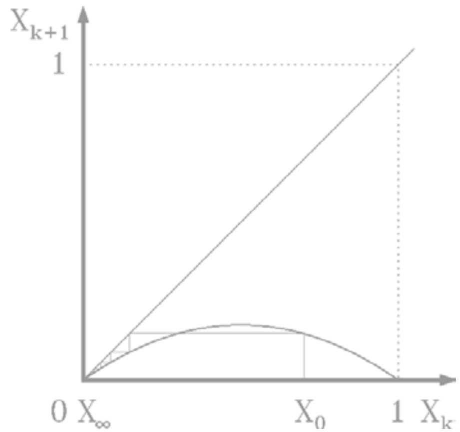

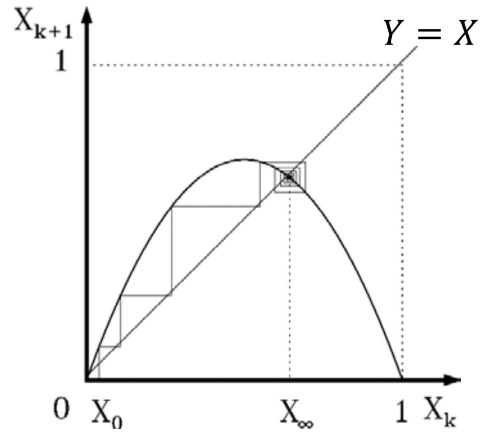

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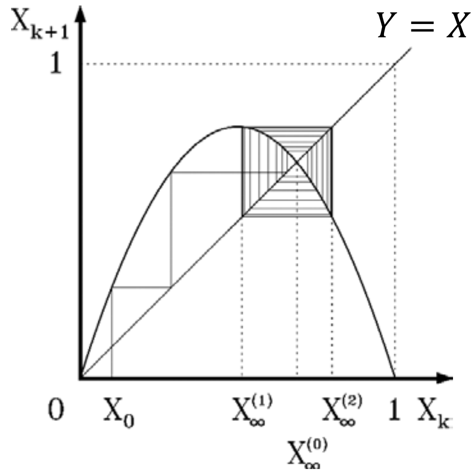
- When the parabola exceeds the line, the ***origin*** always ***repels****.* **(!)**

The logistic dynamics

$\alpha = 3.2$
period 2



• When α is greater than 3, what happened to the origin occurs to the other intersection between the line and the curve: such a location ***repels*** the dynamics and there appear ***repetitions*** every ***two generations***. **(!)**

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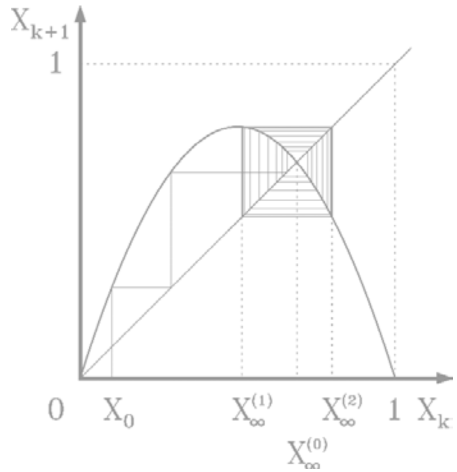

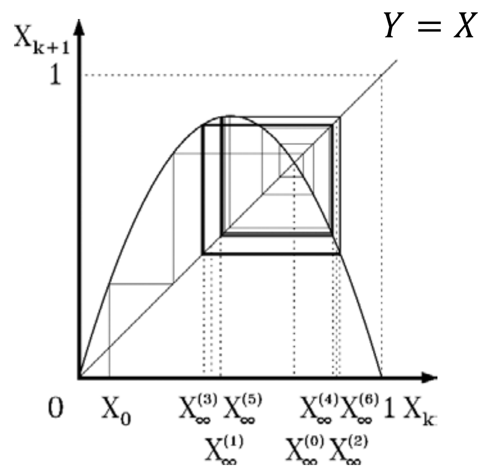

$\alpha = 3.46$
period 4

- When α is greater than 3, what happened to the origin occurs to the other intersection between the line and the curve: such a location ***repels*** the dynamics and there appear ***repetitions*** every ***two generations***. (!)

- If α continues growing, such repetitions ***repel*** and there appear ***repetitions*** every ***four generations***. **(!)**

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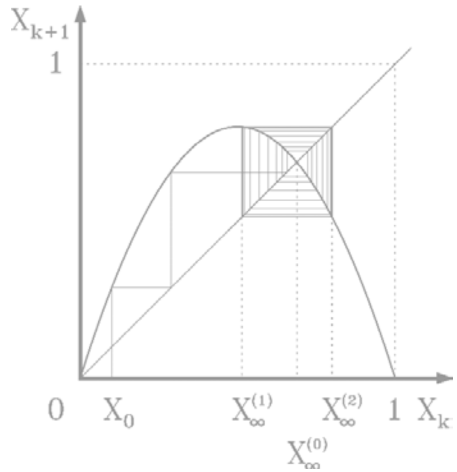

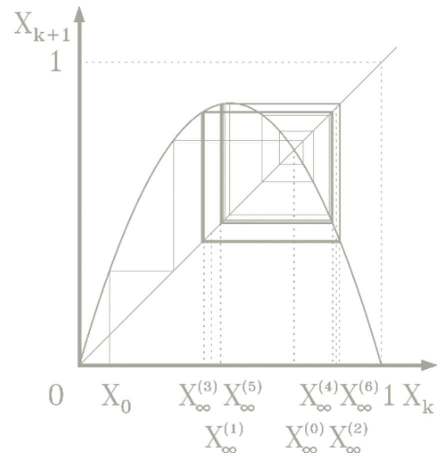

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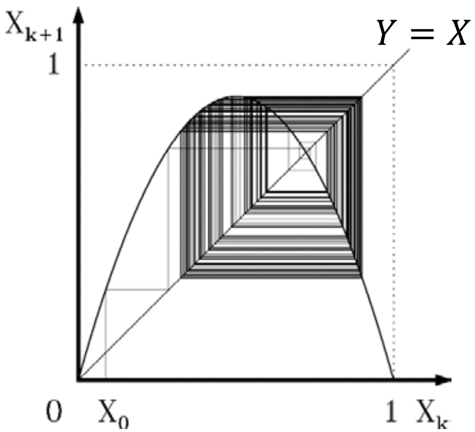
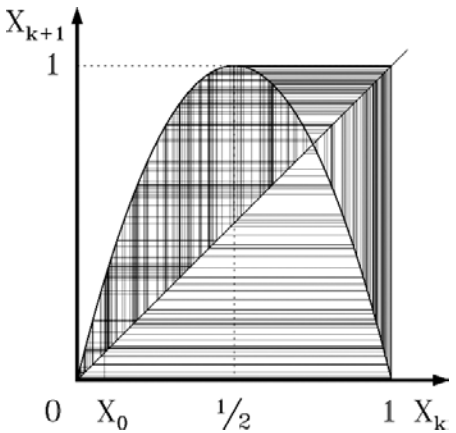
- Surprisingly, there appears a "***chain of bifurcations***": every power of 2 happens before $\alpha_\infty \approx 3.5699\ldots$ **(!)**

The logistic dynamics

$\alpha = 3.6$
aperiodic
strange





$\alpha = 4$
aperiodic
chaotic

- When $\alpha > \alpha_\infty$, there appear infinite "***strange***" attractors exhibiting no repetition, that is, like the expansion of irrational numbers, and they appear as guided by chance, although they are given by a ***deterministic*** process. **(!)**

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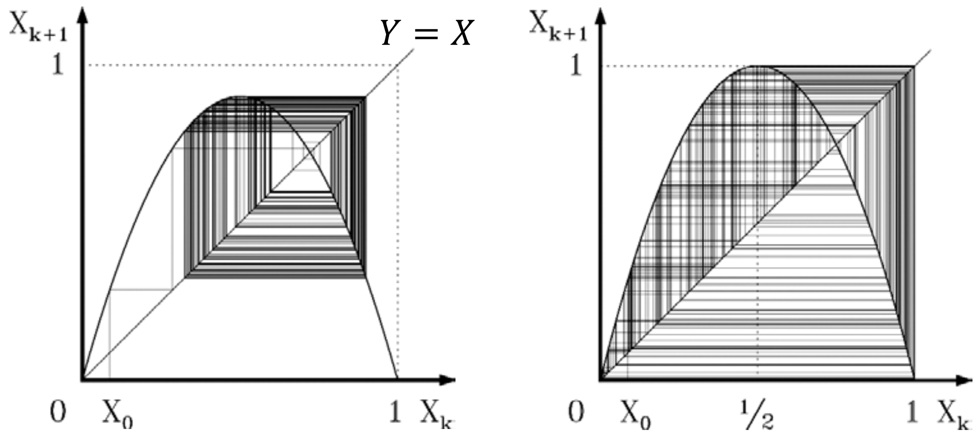

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• As is observed, such sets have the structure of ***dust*** and they define the well-named behavior we call ***chaotic***. **(!)**

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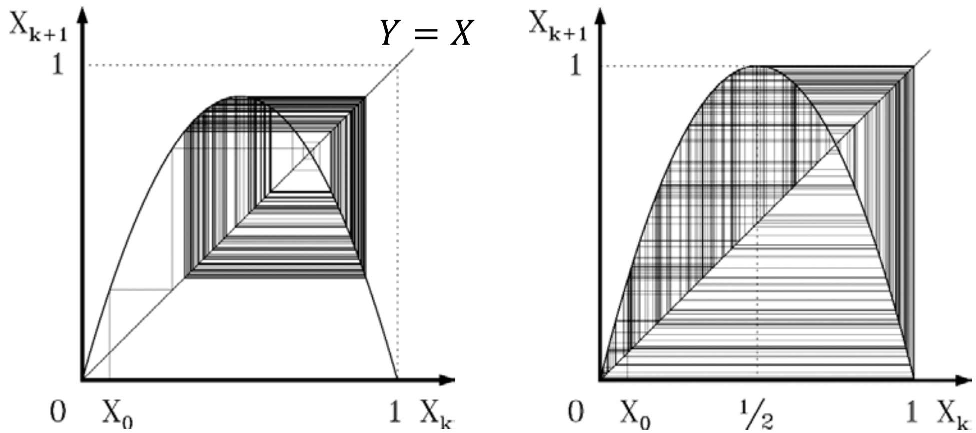

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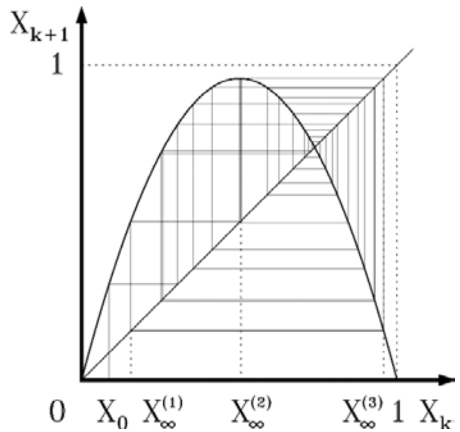
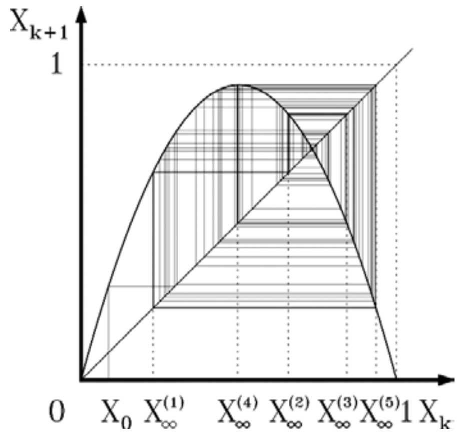
- When $\alpha = 3.6$ the attractor contains two separate zones, but when $\alpha = 4$ the set encompasses almost all the interval from 0 to 1, but with ***small little holes*** as it is ***dusty***.

The logistic dynamics

$\alpha = 3.83$
period 3





$\alpha = 3.74$
period 5

- When α is greater than α_∞, there appear also ***repetitive*** attractors whose repetitions are not powers of 2: for $\alpha = 3.83$ there appear oscillations every ***three generations*** and for $\alpha = 3.74$ there exist every ***five generations***. **(!)**

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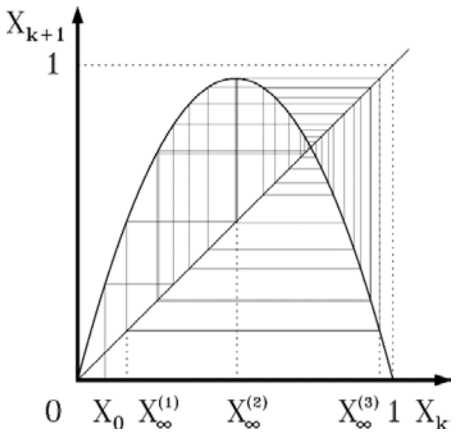

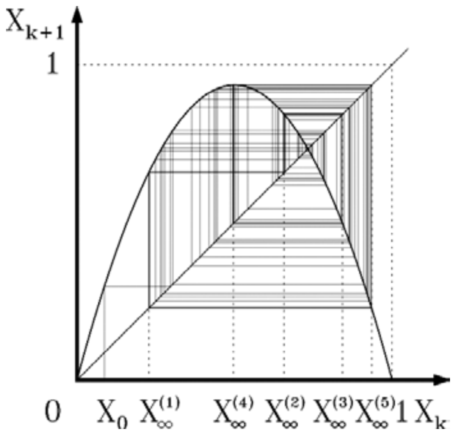

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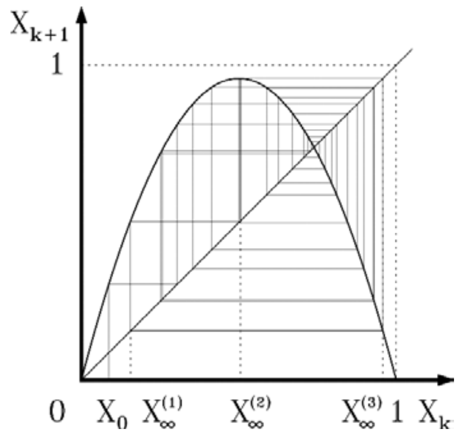

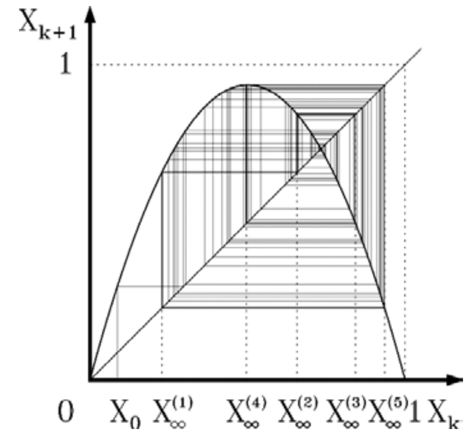

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- As if by magic, the curvatures of the parabolas ***synchronize*** and the horizontal-vertical lines give rise to ***repetitions***. (!)

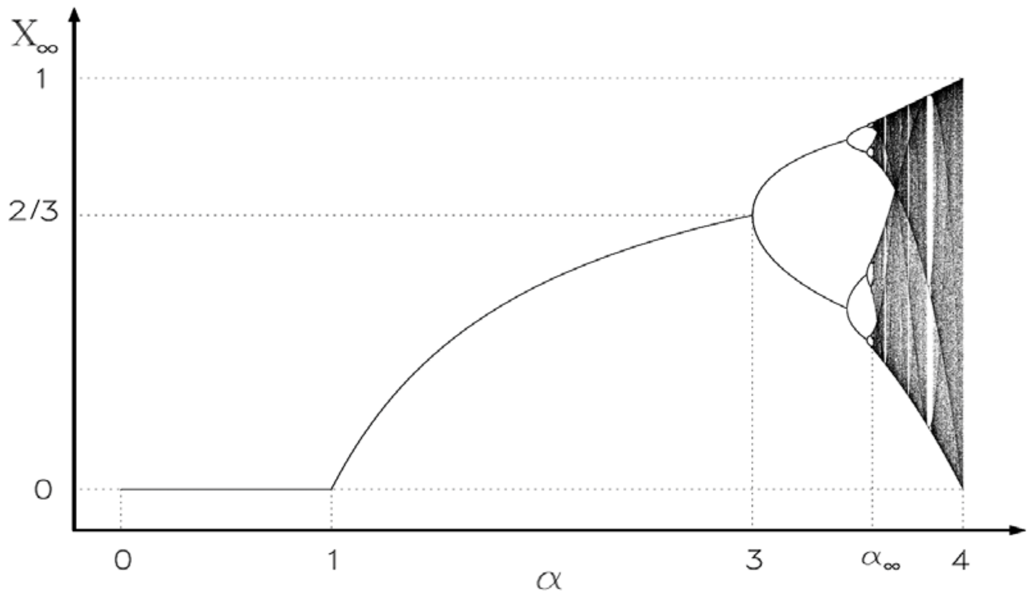
- In an admirable way, the logistic map defines oscillations that correspond to **any** natural number. **(!)**

The diagram of bifurcations

(Feigenbaum, 1978; Maurer and Libchaber, 1979; Puente, 2011, 2019)

The diagram of bifurcations

- X_∞ as a function of α, for stable attractors, is known as the ***diagram of bifurcations***:



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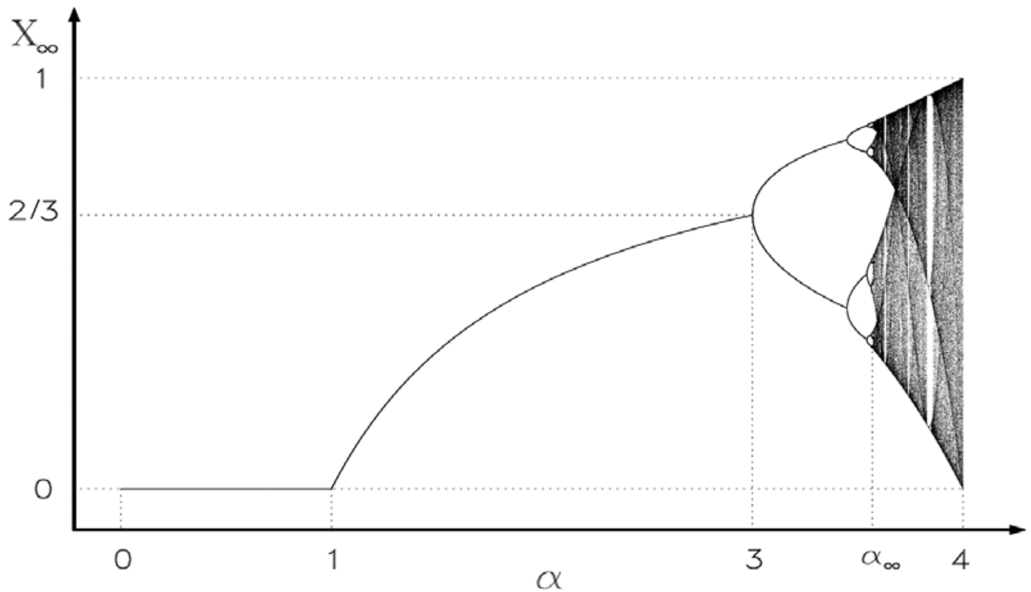

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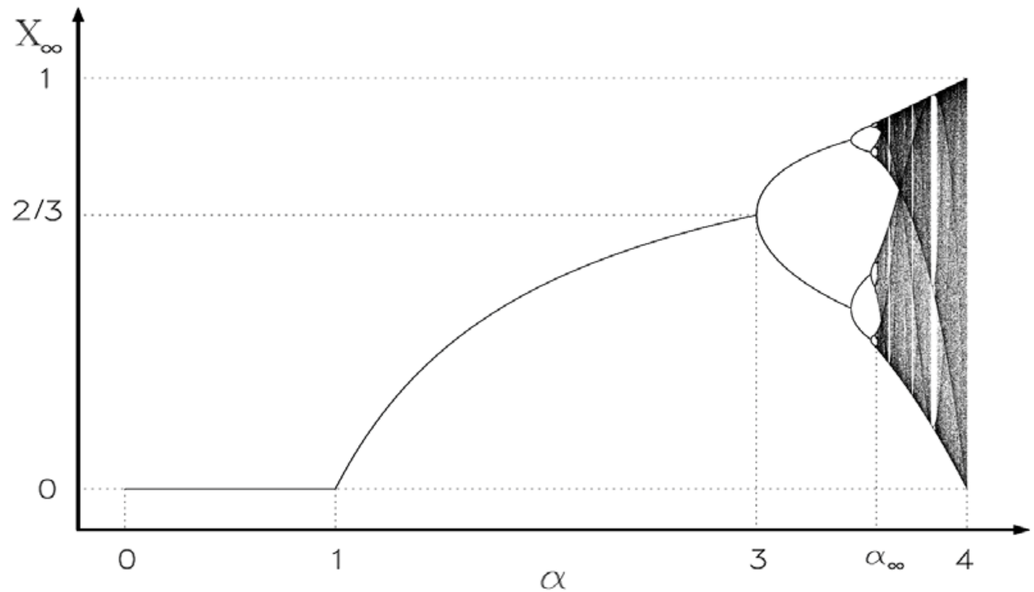

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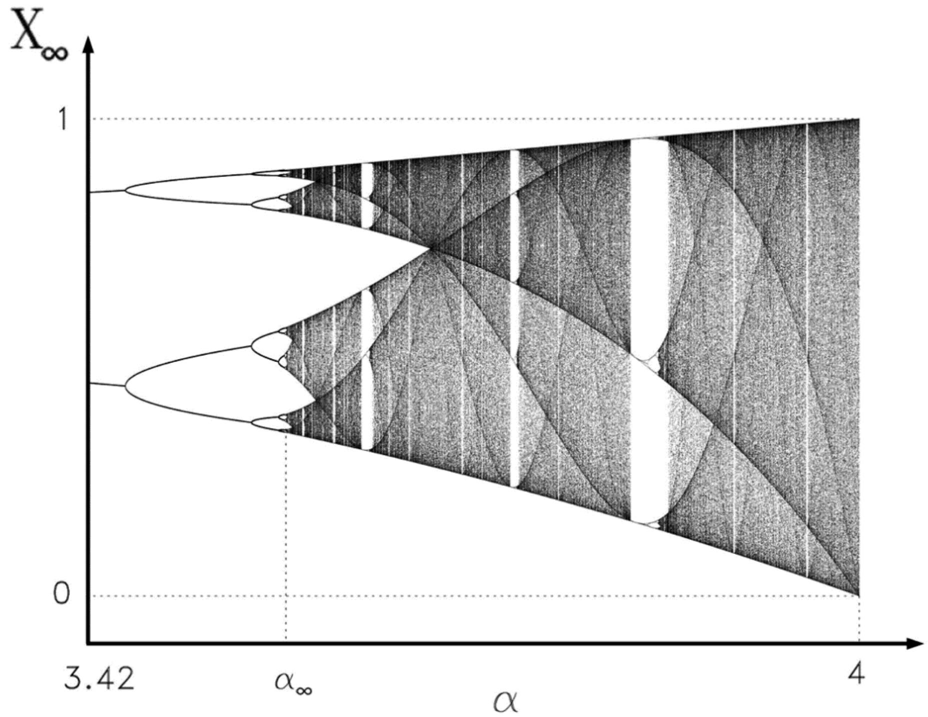
- After α_∞, the ***periodic*** and the ***chaotic*** intertwine, and the infinite strange attractors are little dots in vertical lines.

The diagram of bifurcations

- The striking ***tail*** of the diagram is seen in more detail here:



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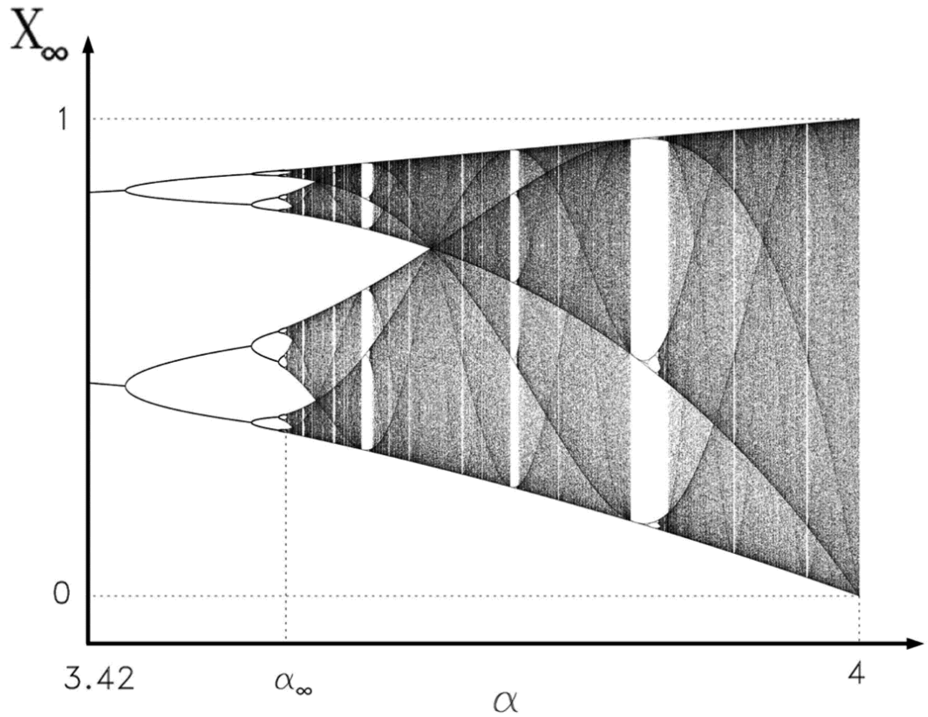
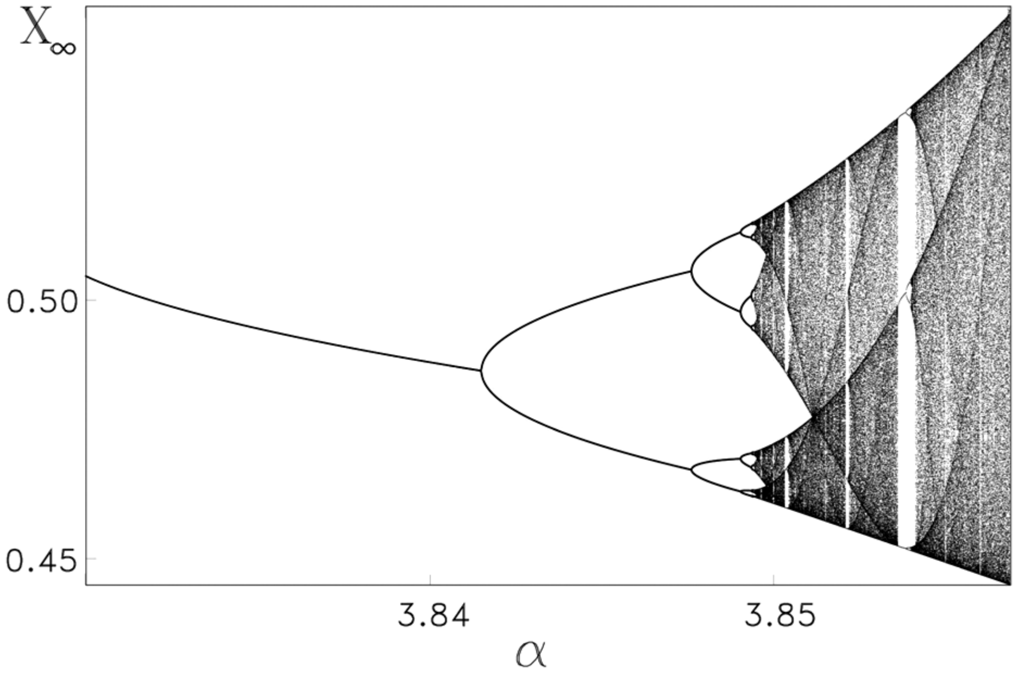


- The "***tree***" contains "***buds***" in periodic "***white bands***" for any value greater than 2, and the most notorious, from right to left, correspond to periods 3, 5 and 6. **(!)**

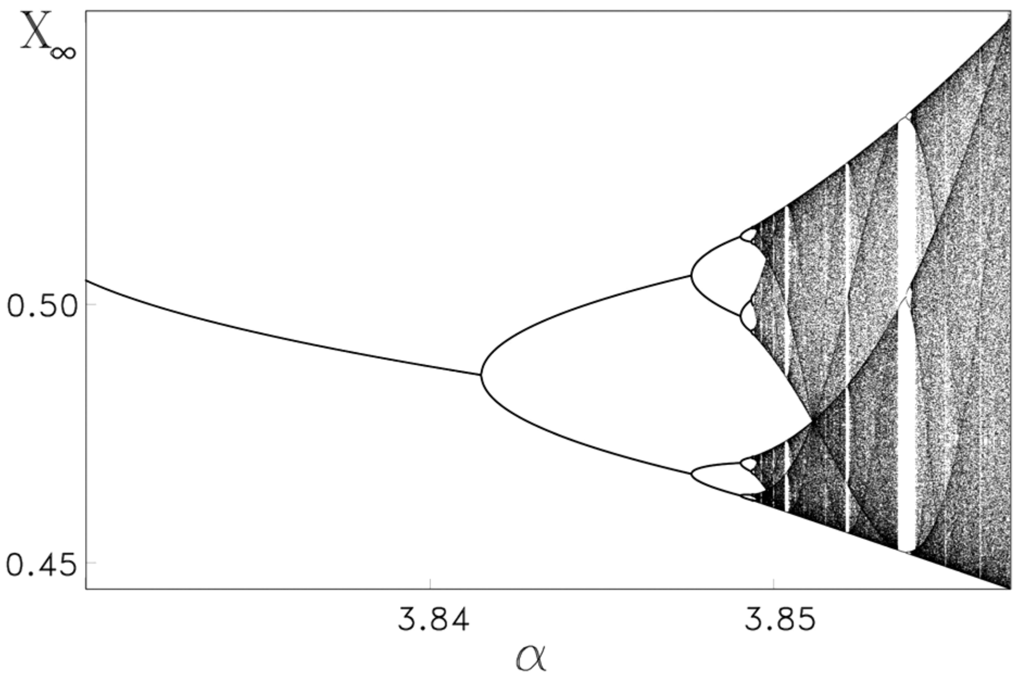
The diagram of bifurcations

- Amplifying the central ***bud*** of period 3 gives:



The diagram of bifurcations

- Al amplificar el ***brote*** central del período 3 resulta:



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The diagram of bifurcations

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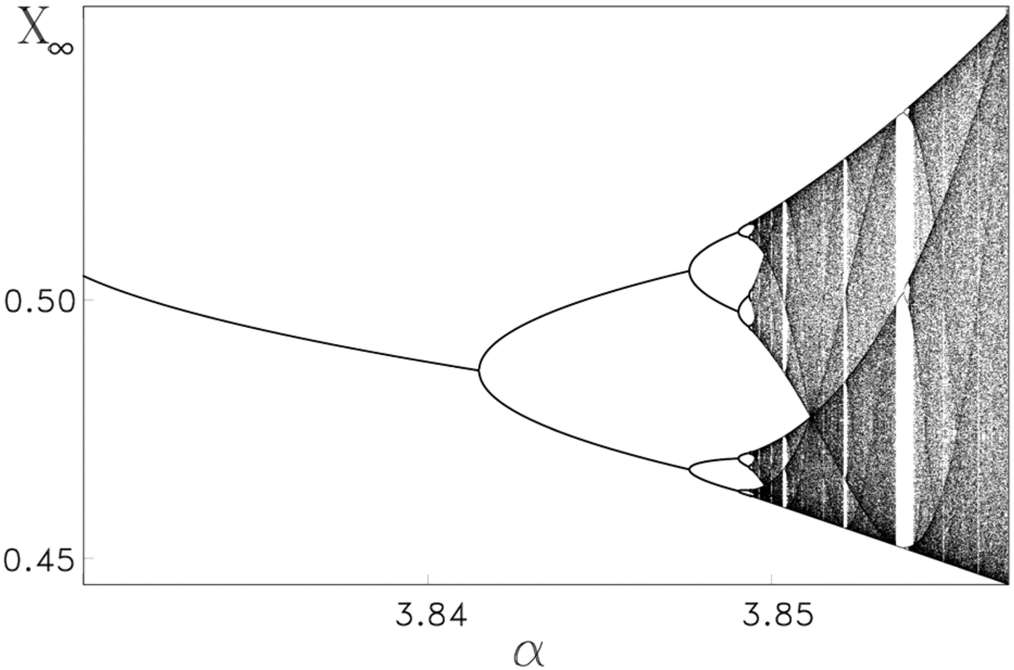
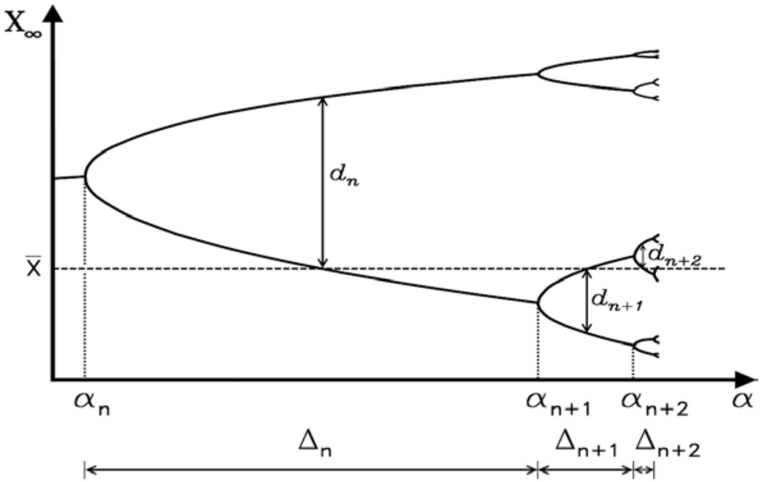


- This is a reduced copy of the ***foliage*** of the ***tree***, without its straight ***root***.

- As the bud contains little buds, the diagram exhibits an exquisite **self-similarity** *ad infinitum.* **(!)**

The diagram of bifurcations

• There exist an ***order*** in this route towards ***chaos***, for, as demonstrated by *Mitchell Feigenbaum* in 1978, all the bifurcations happen according to two ***universal*** constants:



$$d_n/d_{n+1} \longrightarrow \mathcal{F}_1 = -2.50\ ... \qquad \Delta_n/\Delta_{n+1} \longrightarrow \mathcal{F}_2 = 4.66\ ...$$

openings *durations*

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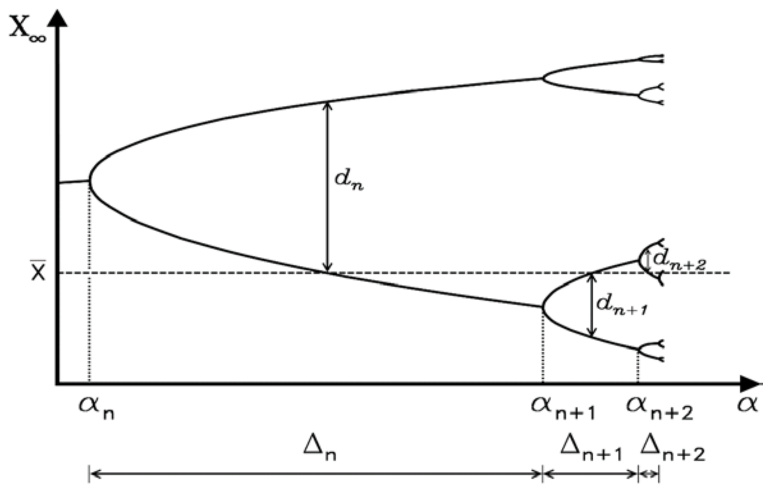

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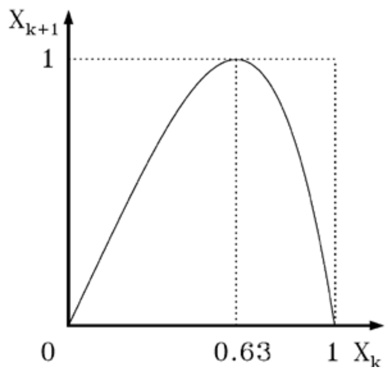
openings *durations*

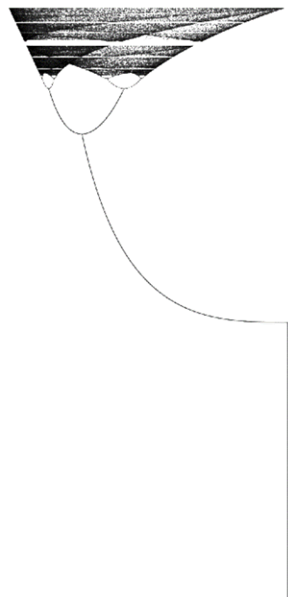
• The diagram of bifurcations is also known as the "***Feigenbaum tree***", or "***the fig tree***", translating from German. **(!)**

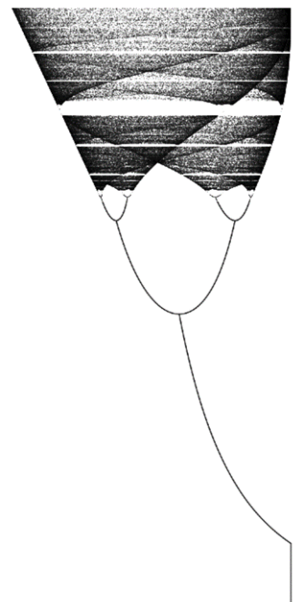
The diagram of bifurcations

• The results are truly ***universal***, as they happen for every curve that has a single peak:

$f(X) = \alpha X(1 - X^3)$







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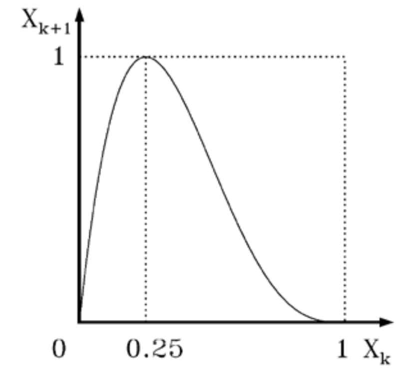

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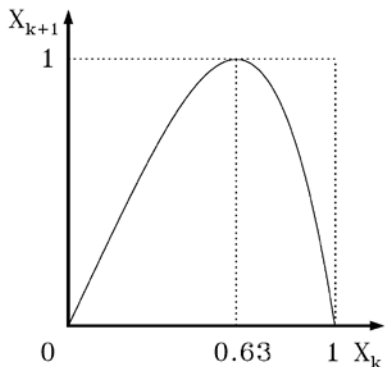

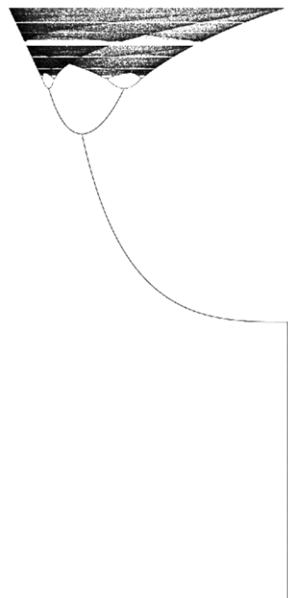

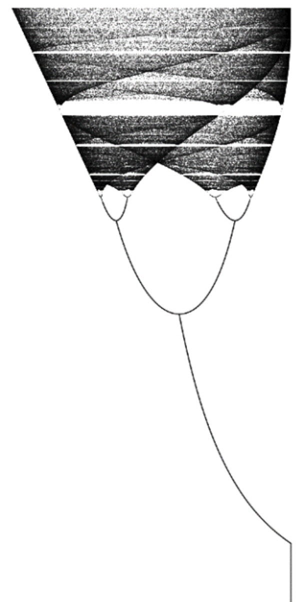

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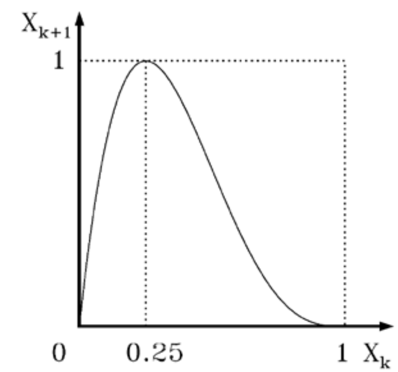

- As per *Feigenbaum*, these trees have a ***straight root***, a "***tender branch***", and ***periodic branches*** intertwined with the ***dust of chaos***.

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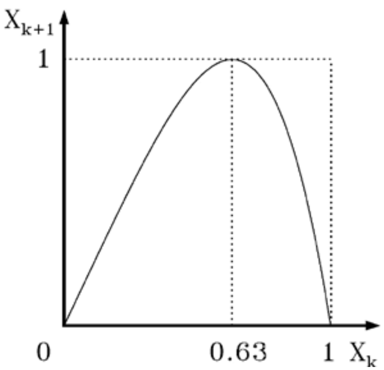

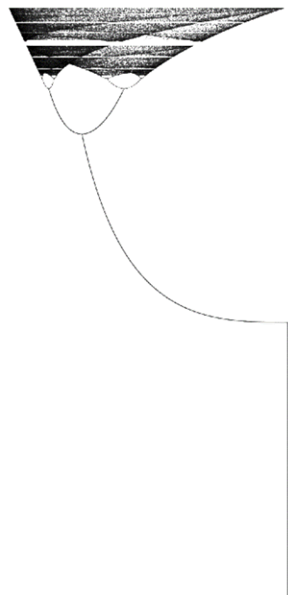

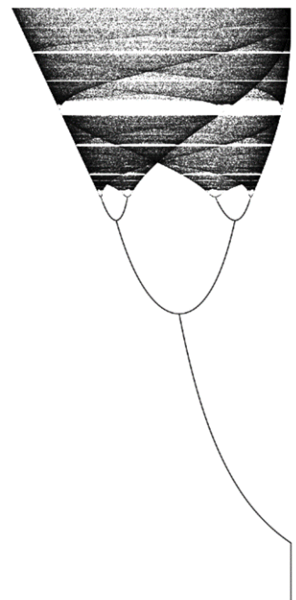

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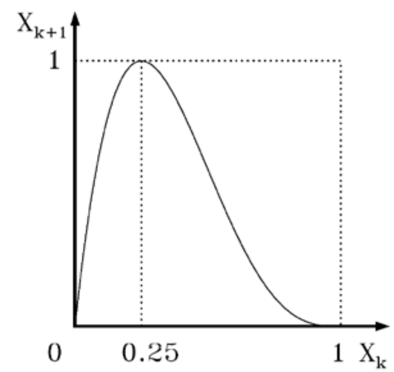

• As per *Feigenbaum*, these trees have a ***straight root***, a "***tender branch***", and ***periodic branches*** intertwined with the ***dust of chaos***.

• These last ones are hence symbolic "***fig leaves***". **(!)**

The diagram of bifurcations

- The results are certainly important, for they are also relevant in ***physics***, ***chemistry***, ***biology***, ***economics***, etc.

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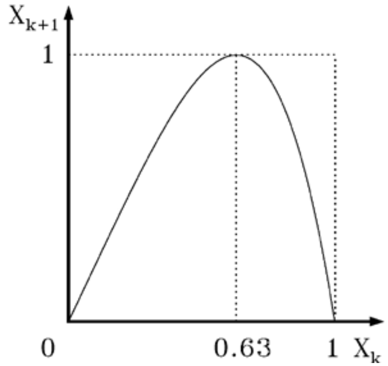

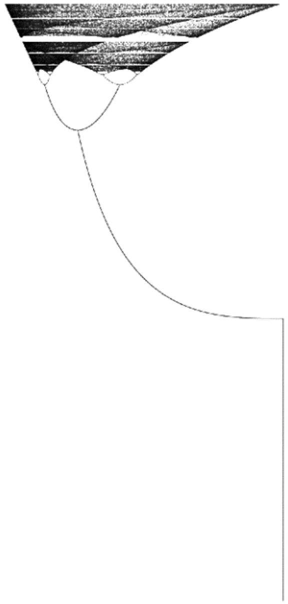

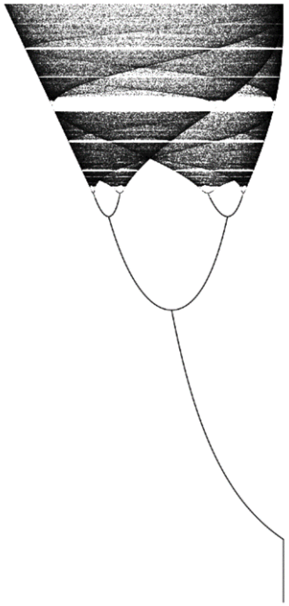

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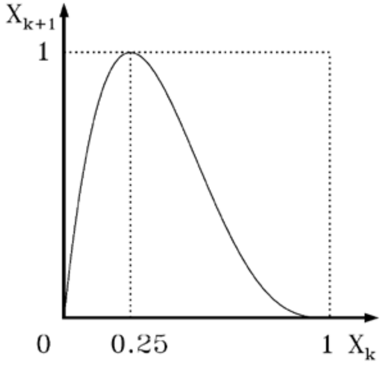

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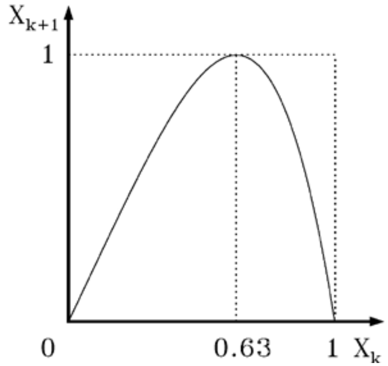

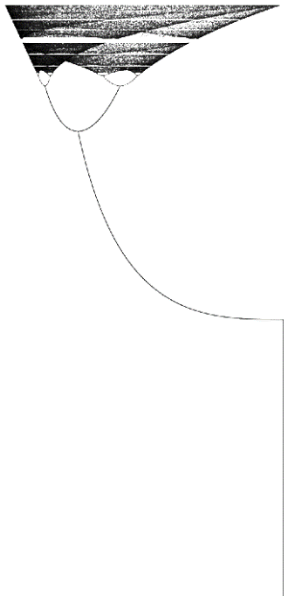

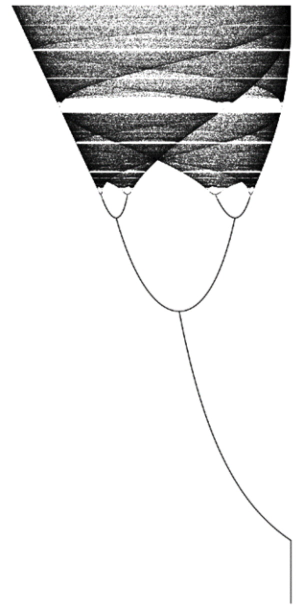

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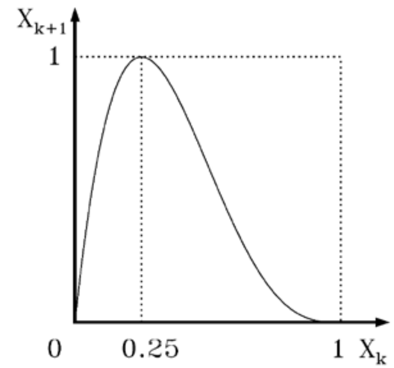

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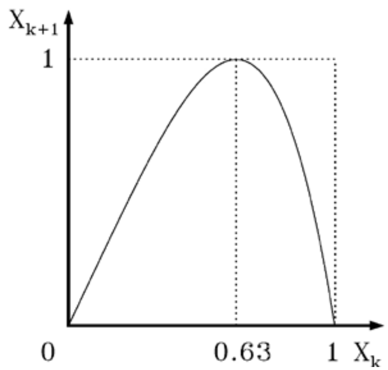

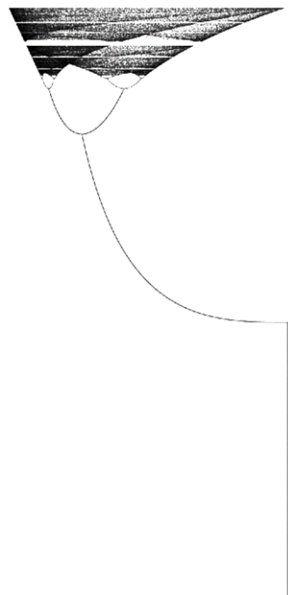

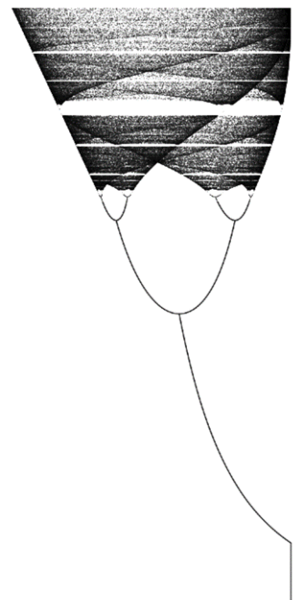

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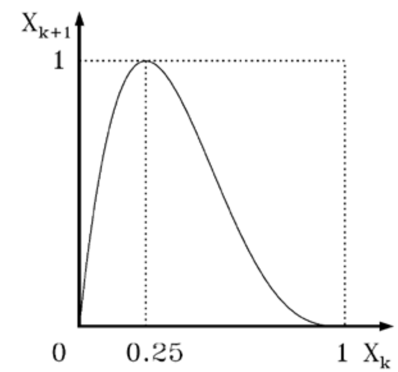
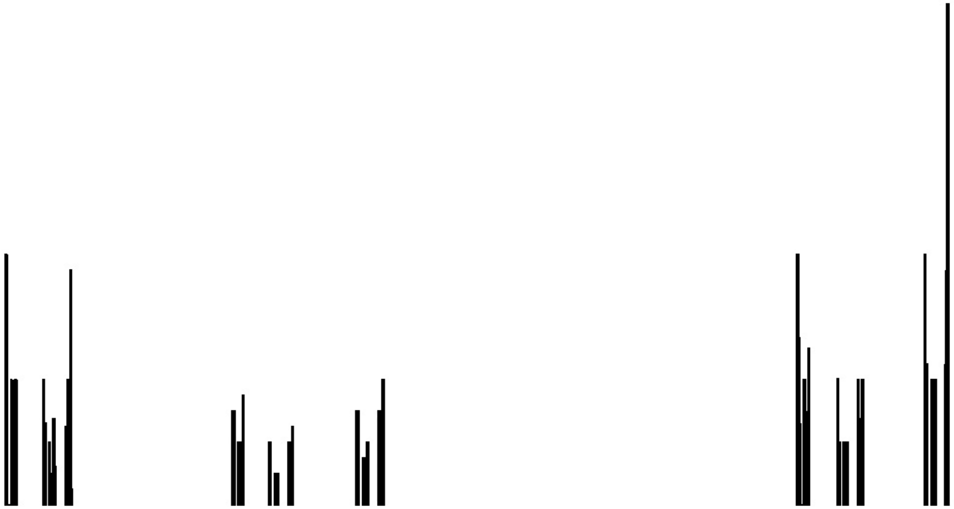


• The dynamics of ***convection*** occur as per *Feigenbaum*, when α denotes the ***heat*** added to a fluid. (!)

• This is so for ***liquid helium***, ***mercury*** and ***water***, as found first by *Jens Maurer* and *Albert Libchaber* in 1979.

The diagram of bifurcations

- The chaotic tree contains sets of ***multi-fractal*** **thorns** that combine ***imbalances*** and ***holes*** as in the previous games related to the study of ***turbulence***. The first one occurs for the value of α_∞:



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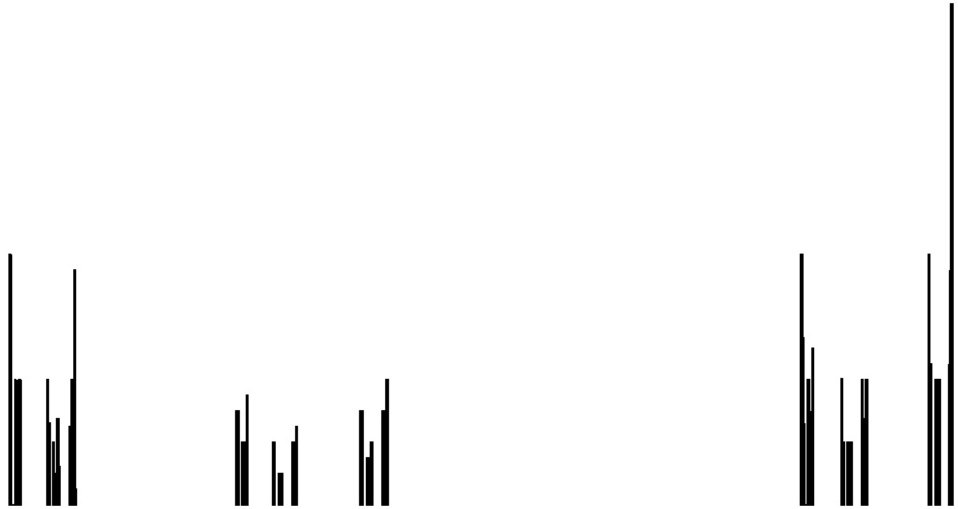

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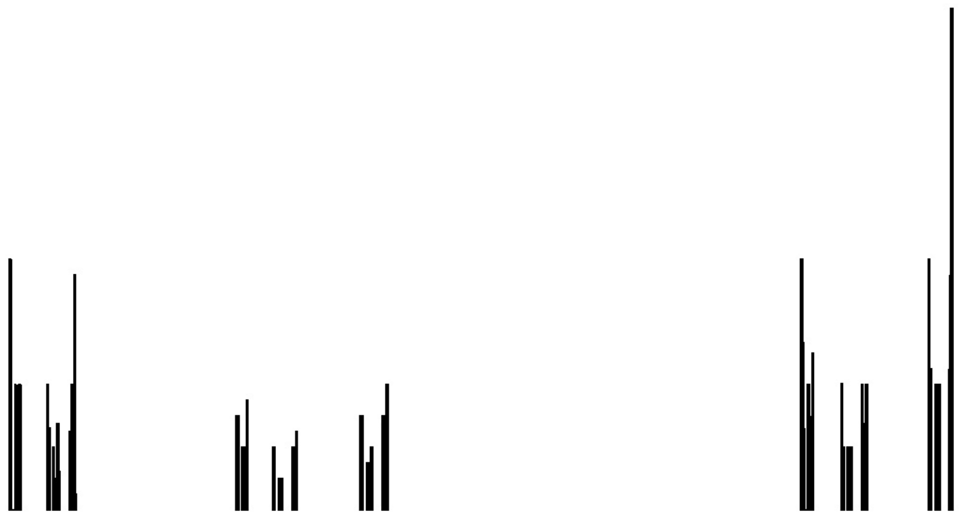

- Such come from a ***histogram*** of the dynamics at such a value.

- The tree is a ***thorn bush***, as there are many ***spikes*** by the end of the white bands of the tree, where the buds define Cantor ***dusts***. **(!)**

Properties of chaos

(Moon, 1987; Peitgen et al., 1992)

The geometry of the strange

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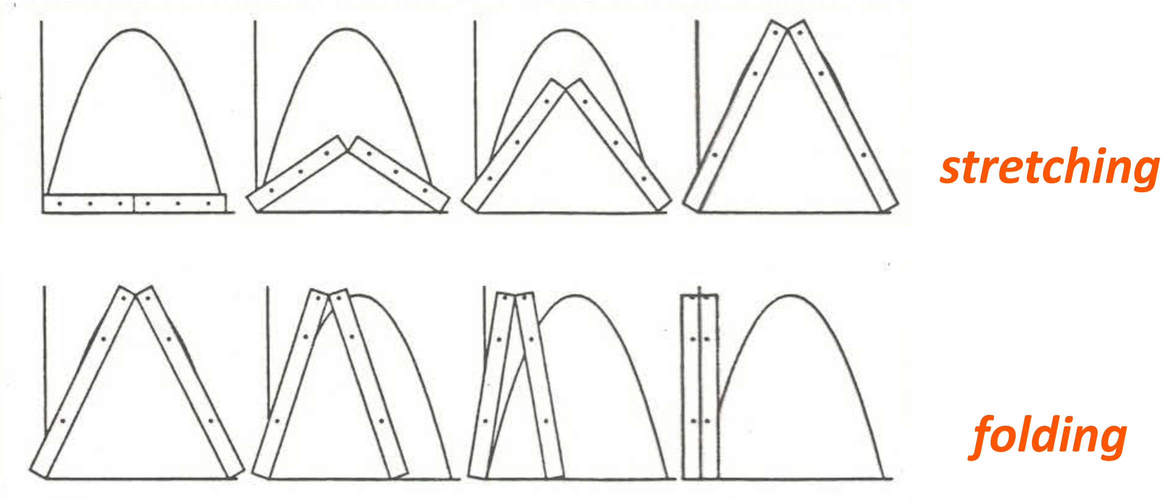
- For the logistic map when $\alpha = 4$ the two steps are:



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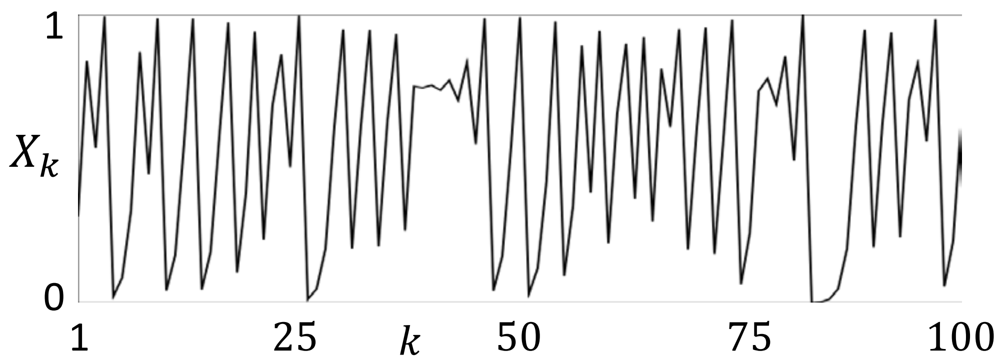

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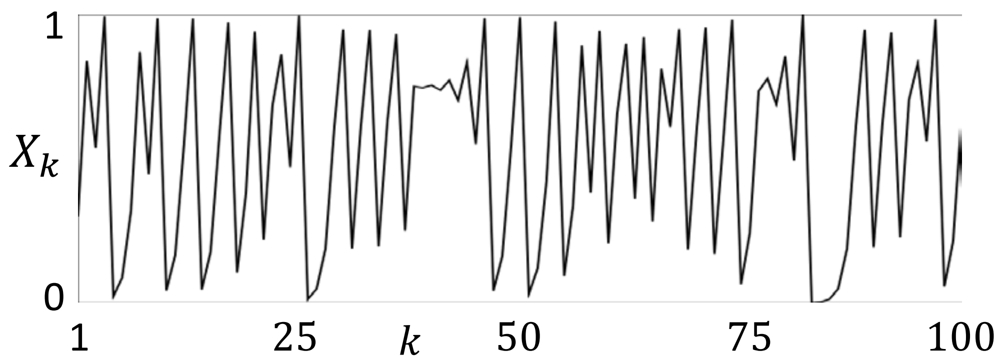

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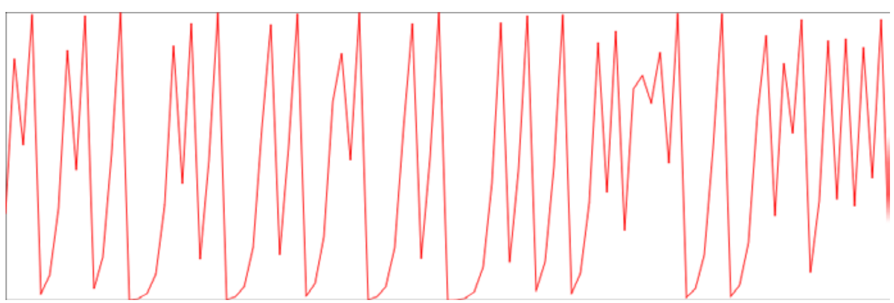

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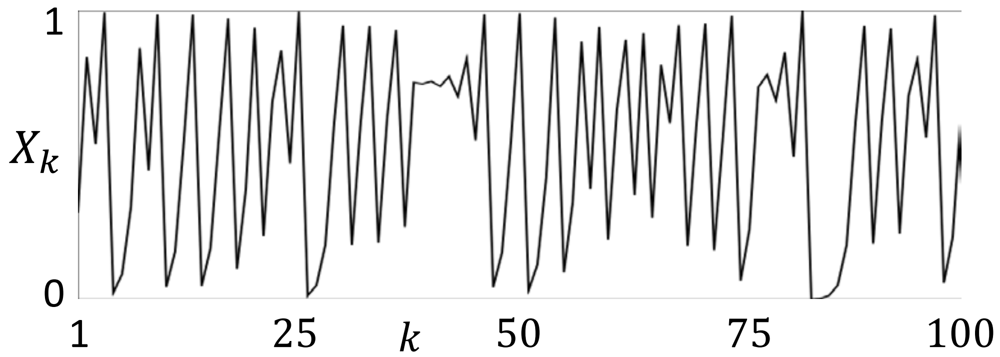

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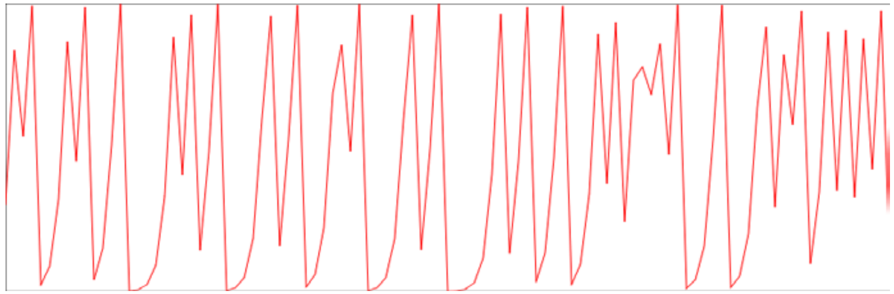

- This is the "***butterfly effect***", a ***divergence*** that prevents us to predict.

The Lyapunov exponent

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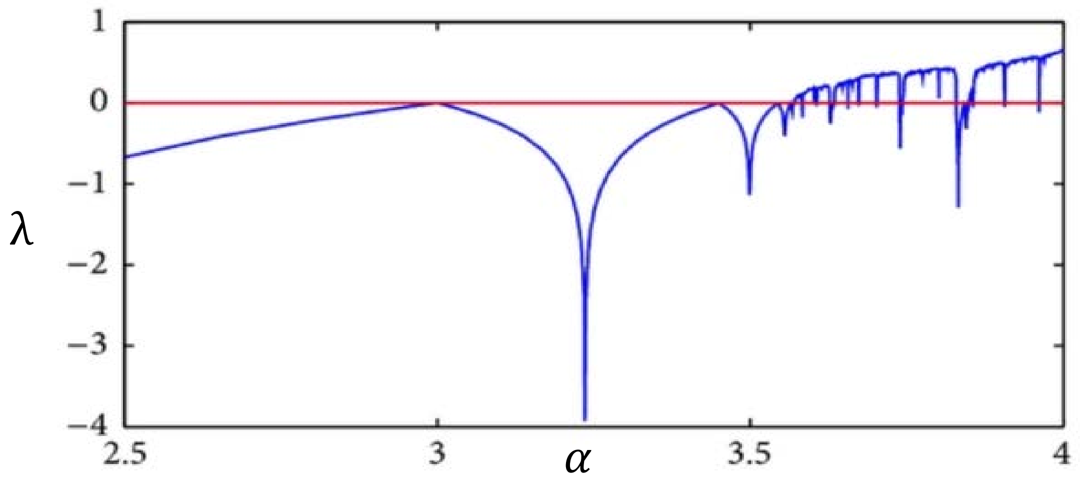

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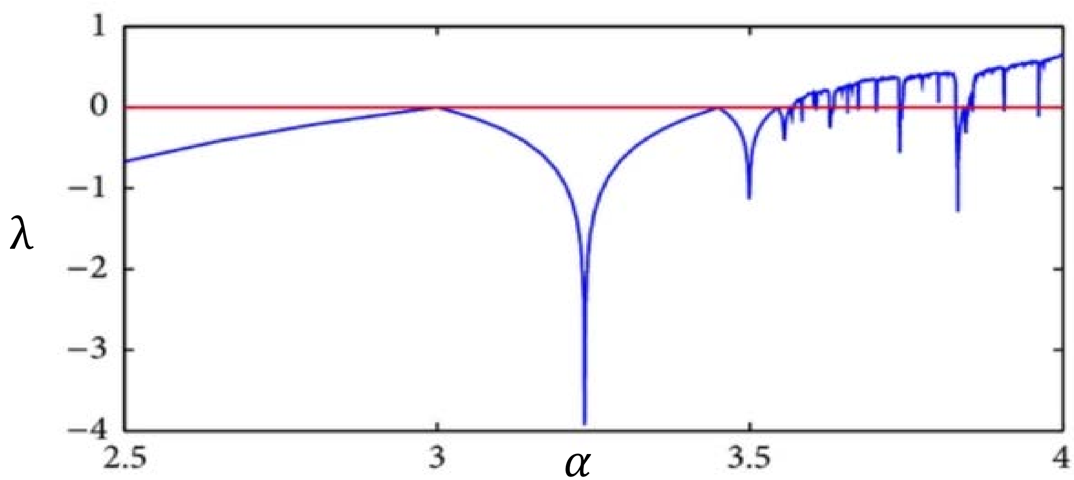


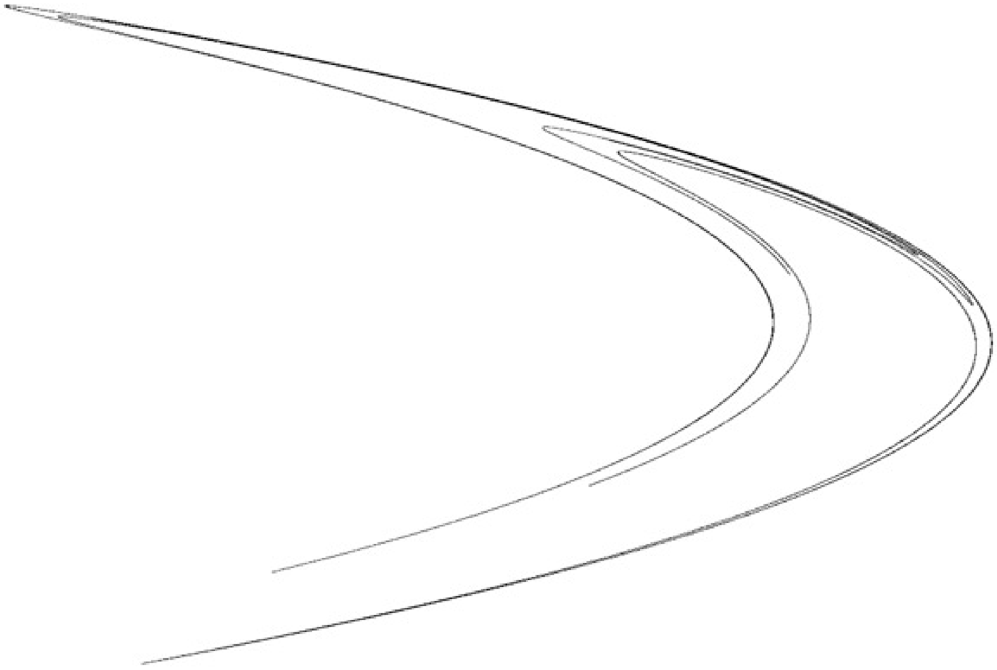
- The maximum value, $\lambda = \ln 2$, occurs at the highest ***heat*** when $\alpha = 4$.

Attractors in 2D and 3D

(Lorenz, 1983; Moon, 1987)

The Hénon attractor

- If the coupled equations, $x_{k+1} = 1 - a x_k^2 + y_k$ and $y_{k+1} = \mathrm{b}\, x_k$ with parameters $a = 1.4$ and $\mathrm{b} = 0.3$, are used, there appears a strange attractor:



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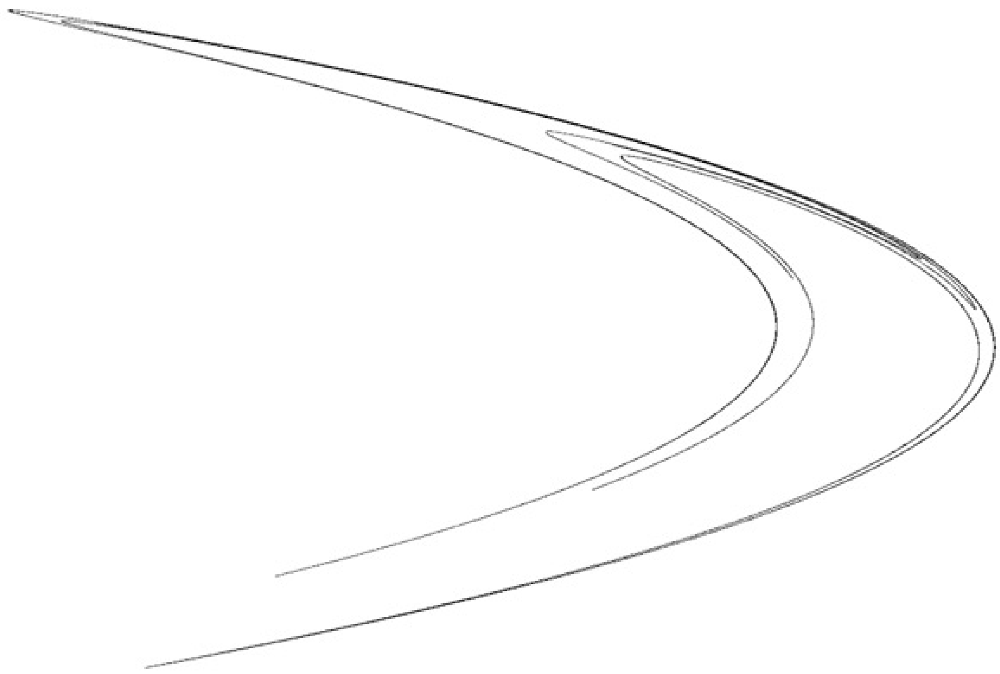
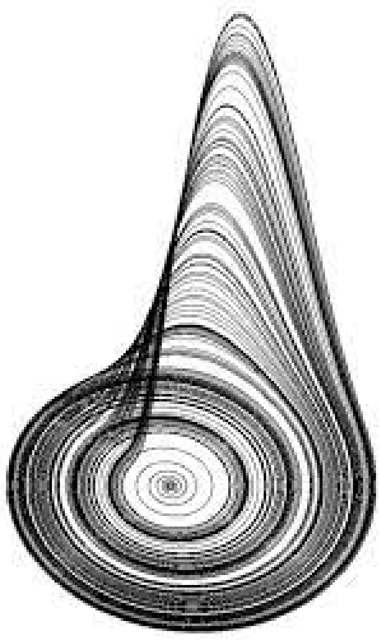


• It looks like a ***napoleon cake*** with a ***Cantorian*** structure, and it has a fractal dimension of 1.26. **(!)**

The Rössler attractor

- If now are employed three coupled equations, but not of differences but differential, $\dot{x} = -\mathrm{y} - z, \dot{y} = x + ay$ and $\dot{z} = b + z(x - c)$, with parameters $a = 0.2$, $b = 0.2$ and $c = 5.7$, there appears a strange attractor in 3D:



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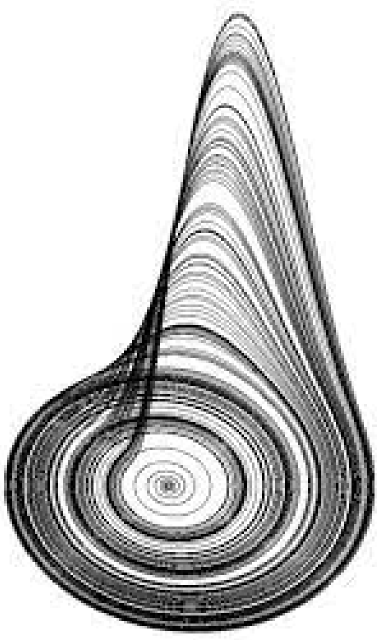
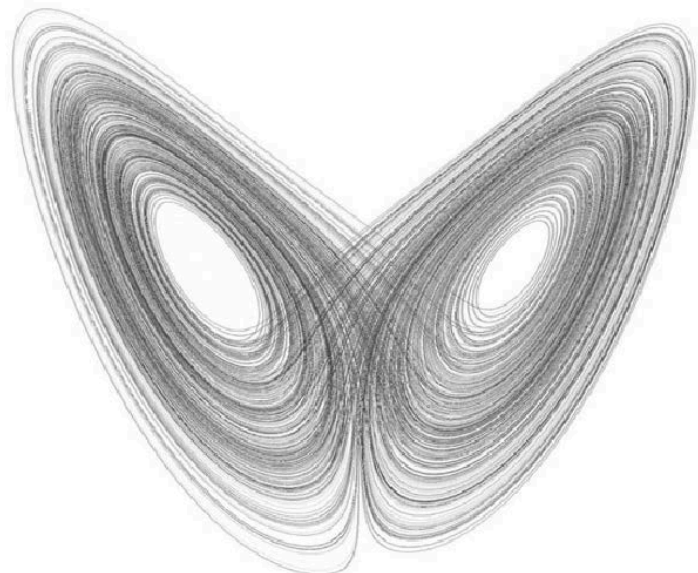


- This ***Cantorian*** object has the structure of a ***Möebius strip***.

The Lorenz attractor

- The three equations, $\dot{x} = \sigma(\mathrm{y} - z), \dot{y} = x(\rho - z) - y$ and $\dot{z} = xy - \beta z$, with parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$ generate:



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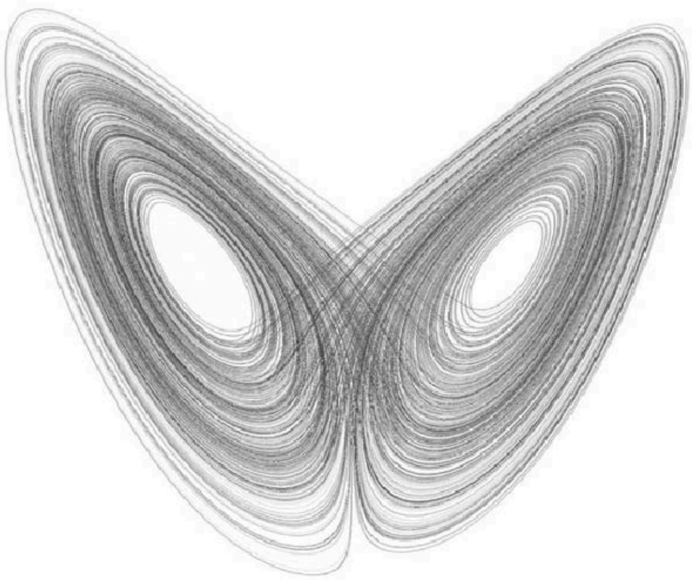

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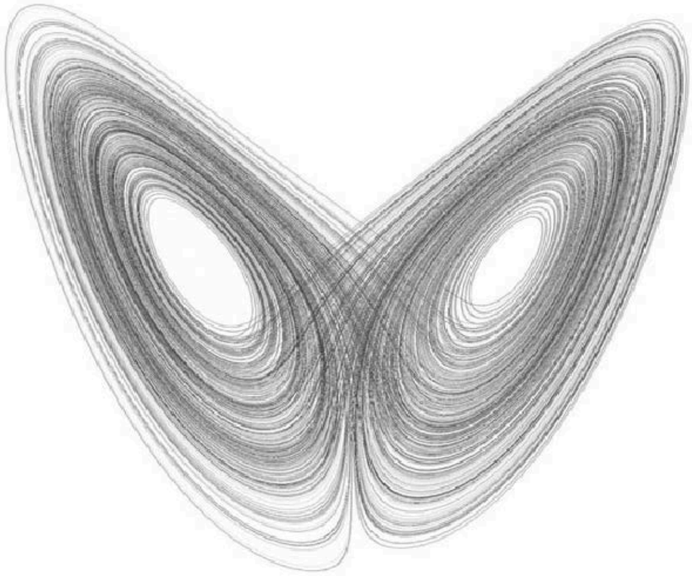

• The object looks like a "butterfly" and its dimension is 2.06. (!)

• These equations, used by *Edward Lorenz* to study climate, allowed him to identify, for the first time in 1963, the ***butterfly effect*** of deterministic, aperiodic strange attractors.

…Well, here ends this brief introduction.

In our next encounter we shall see, based on these ideas, how we may comprehend that Jesus is the narrow gate and the only way to the Father.

Until next time…

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