Chaos, Complexity & Christianity

2. An introduction to fractals and complexity

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Summary

- *Recalls the different kinds of numbers: naturals, integers, rationals and reals.*
- *Reviews the concept of dimension for points, lines, planes and volumes.*
- *Shows examples of fractal objects, including Cantor dust, the Koch curve and the Sierpinski triangle.*
- *Contrasts order with chaos via the logistic map.*
- *Introduces power-laws related to natural complexity.*

Let’s talk about numbers

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- **Infinity** is certainly an odd concept, for we have shown that

$$2 \cdot \infty + 1 = \infty \quad \textbf{(!)}$$

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- Sometimes such *"**steady state**"* appears immediately, as in 2/3 and 1/11, or it is reached after a *finite* *"**transient state**"*, as it happens with 1/2.

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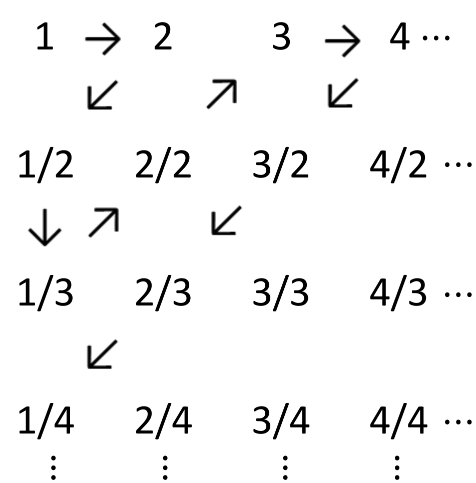

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$$
\begin{array}{ccccccc}
1 & \rightarrow & 2 & & 3 & \rightarrow & 4 \cdots \\
 & \swarrow & & \nearrow & & \swarrow & \\
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\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

- **Infinity** has, in truth, its own rules: $\infty \cdot \infty = \infty$ **(!)**

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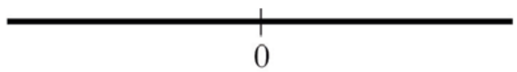

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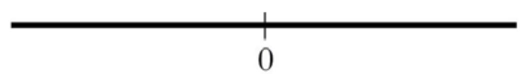

- These cannot be ***counted*** and belong to a ***larger infinity***. (!)
- For, if we assume there exists a list:

$$\begin{array}{ll} 1st & 0.a_1a_2a_3a_4a_5\ldots \\ 2nd & 0.b_1b_2b_3b_4b_5\ldots \\ & \vdots \\ nth & 0.x_1x_2x_3x_4x_5\ldots \end{array}$$

then $0.y_1y_2y_3y_4y_5\ldots$ is not in the list if

$y_1 \neq a_1, y_2 \neq b_2, \ldots, y_n \neq x_n$, etc., which results in a contradiction. **(!)**

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- Fully understanding the irrationals is not possible unless they possess a particularly defining property, like the famous numbers above.

The concept of dimension

(Mandelbrot, 1982)

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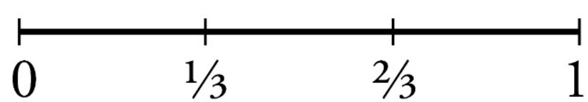

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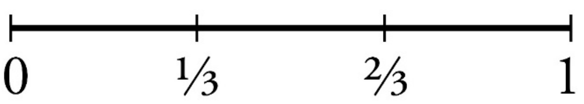

- If $\delta = 1/n$, $N(\delta) = n$, and then $N(\delta) = \delta^{-1}$, which defines the **dimension** of the ***straight line*** as the negative of the exponent, or $D = 1$. **(!)**

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- As the ***plane*** contains ***infinite lines*** and ***points***, $\infty \cdot 1 = 2$ and $\infty \cdot 0 = 2$. **(!)**

Some fractal sets

(Mandelbrot, 1982; Barnsley, 1988; Feder, 1988)

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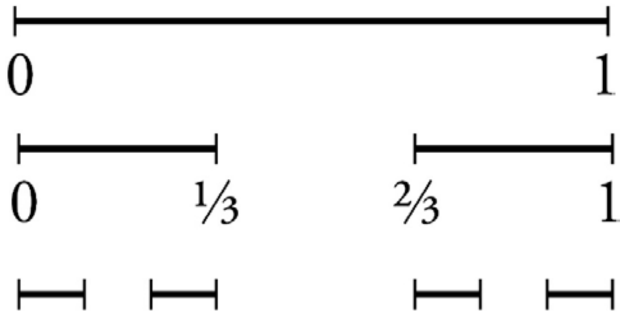
- Consider what is left after taking away ***successive open intervals of size a third and by the middle*** from an interval of size 1:



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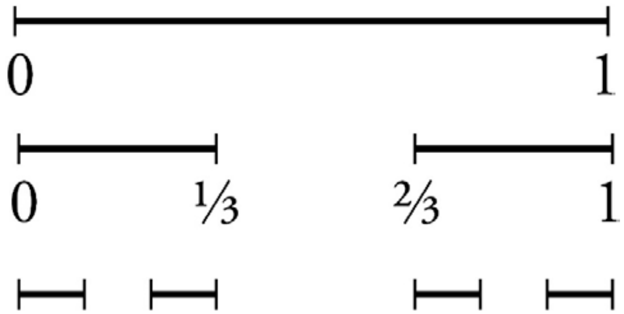

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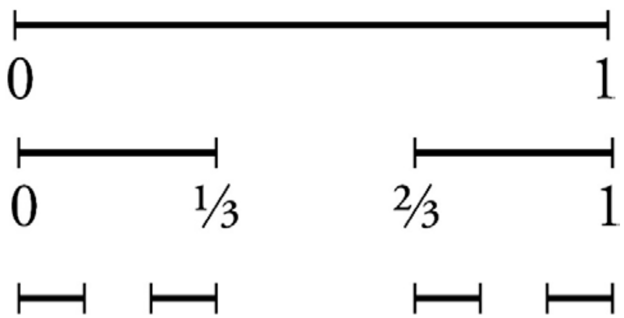

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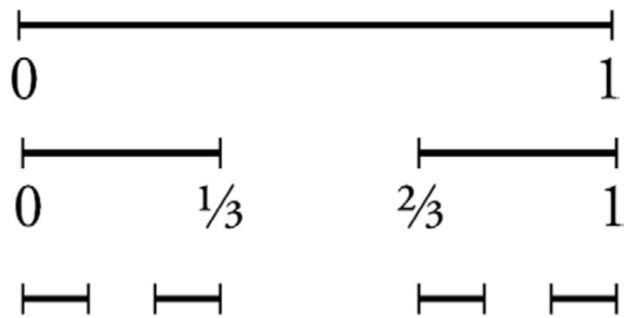

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• As $\mathrm{N}(1/3) = 2$, $\mathrm{N}(1/9) = 4$; $\mathrm{D} = \ln 2/\ln 3 \approx 0.63$, thus $\infty \cdot 0 \approx 0.63$. **(!)**

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δ	$N(\delta)$
1	1
$1/3$	2
$1/9$	4
$\vdots$	$\vdots$
$1/3^n$	2^n

$$\delta = 1/3^n \Rightarrow \ln\delta = -n\ln 3$$

$$n = -\frac{\ln\delta}{\ln 3}$$

Then,

$$N(\delta) = 2^n$$

$$= 2^{-\frac{\ln\delta}{\ln 3}}$$

$$= e^{-\frac{\ln\delta\ \ln 2}{\ln 3}}$$

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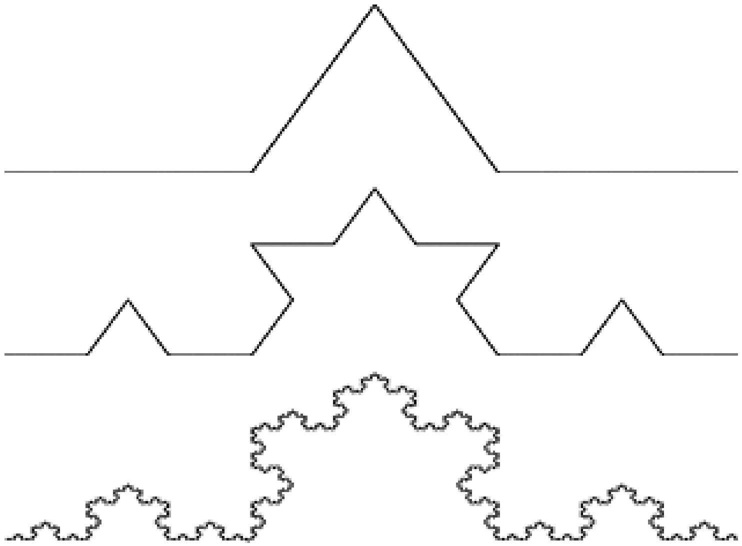
- There are other **fractals** defined over two and three dimensions.

The Koch curve

- Consider the set found replacing each line segment by ***four smaller segments having a third of a size and making up, by the middle, an equilateral triangle***:



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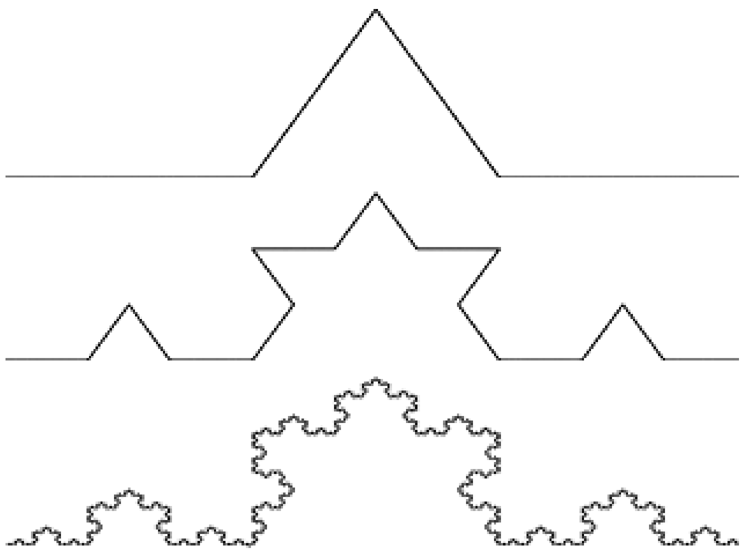

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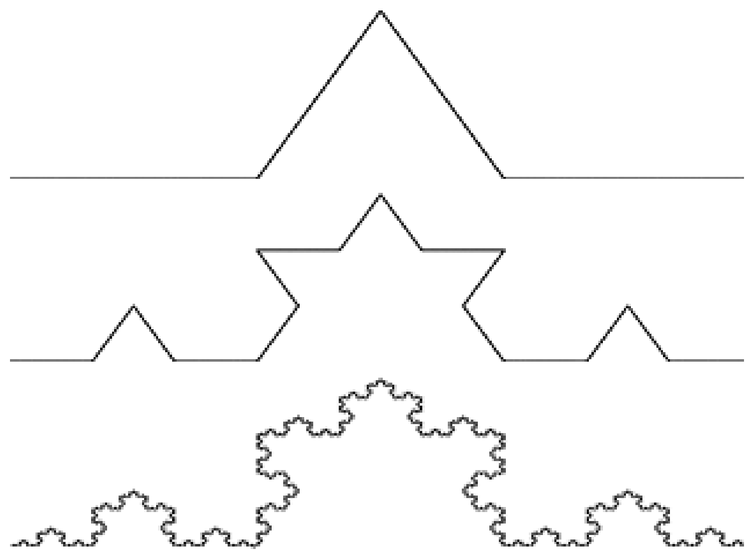

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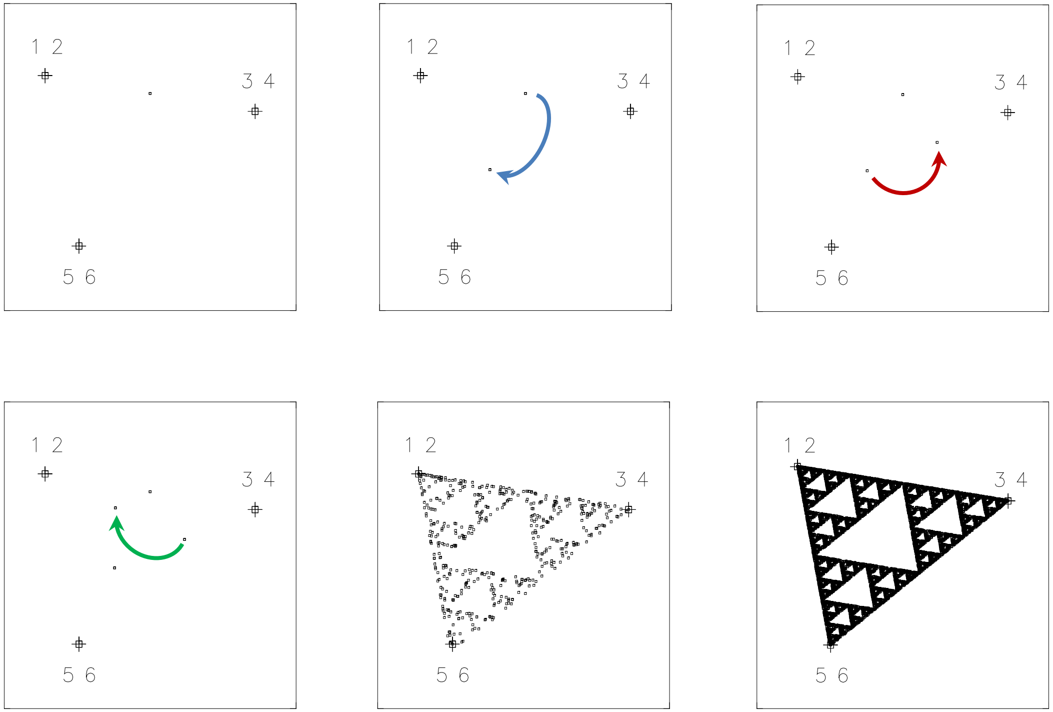
• There are other such sets with dimensions between 1 and 2 (inclusive).

The Sierpinski triangle

- Obtained ***dot by dot*** going towards the middle of three vertex or ***taking away central triangles successively***, as introduced in 1915:



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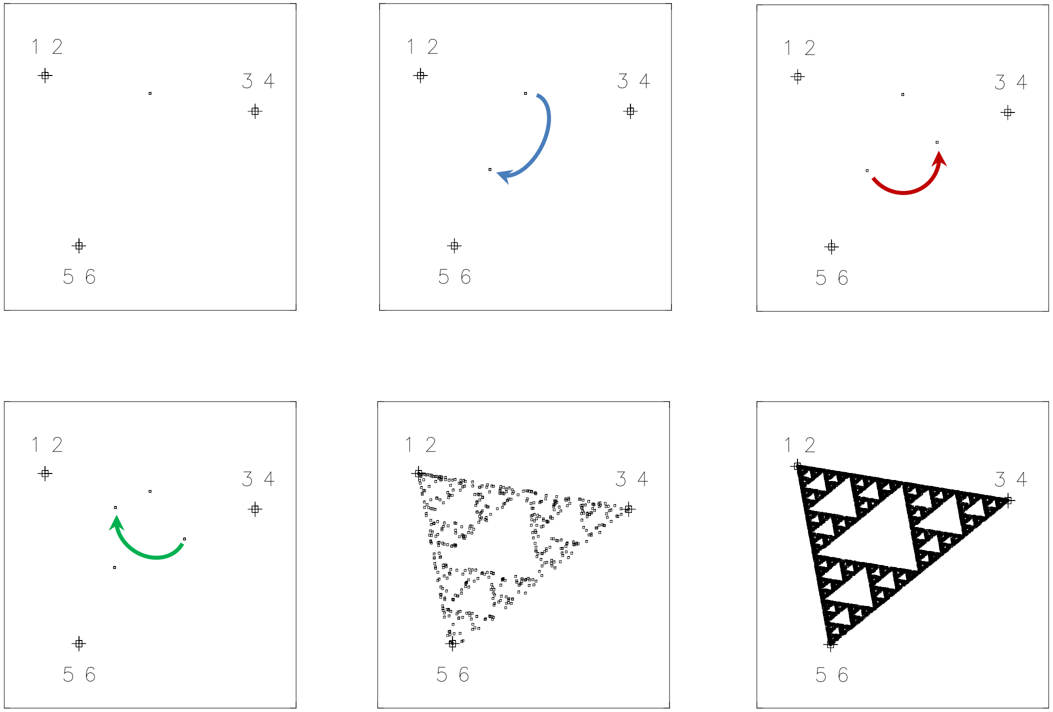

- This is another **fractal** set, $D = \ln 3/\ln 2 \approx 1.58$, and here $\infty \cdot 0 \approx 1.58$. **(!)**

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• Fractals are relevant in physics, geophysics, economics, biology, etc.

• In fact, fractals are everywhere and their implicit repetitiveness, their "***self-similarity***", is reflected in a ***simple*** **power-law**:

$$N(\delta) \sim \delta^{-D}$$

Order and chaos

(Lorenz, 1963; May, 1976; Gleick, 1987)

Order and chaos

- **Fractal** sets are also found in the dynamics of ***non-linear*** systems. To illustrate this, it is pertinent to study the *quadratic* **logistic map**:

$$X_{k+1} = \alpha X_k (1 - X_k)$$

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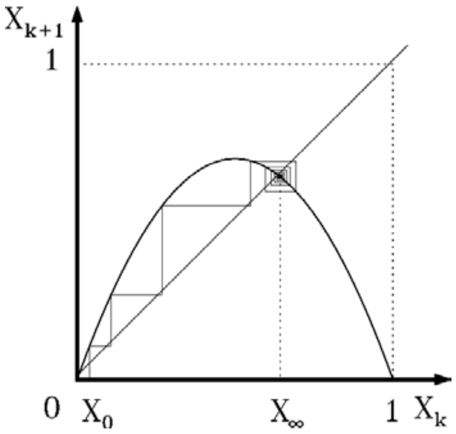
• When $\alpha = 2.8$, a ***stable*** population, X_∞, appears, defining ***order***:



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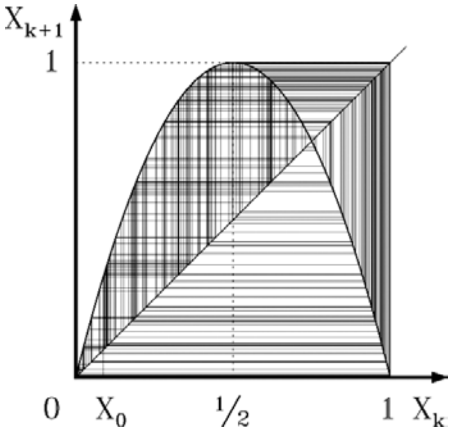
- When $\alpha = 4$ the population does not rest at all, but *wanders in* ***dust*** *forever* on a ***fractal*** *attractor*:



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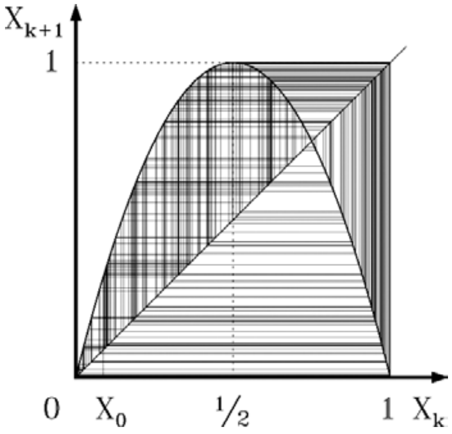

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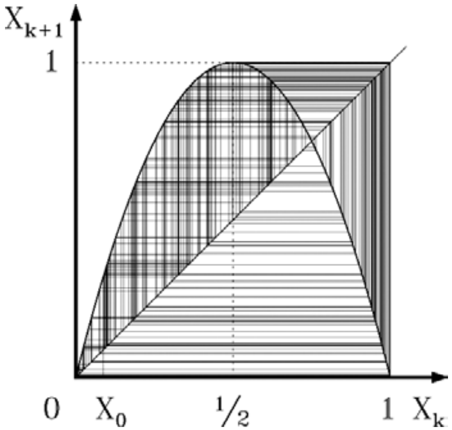

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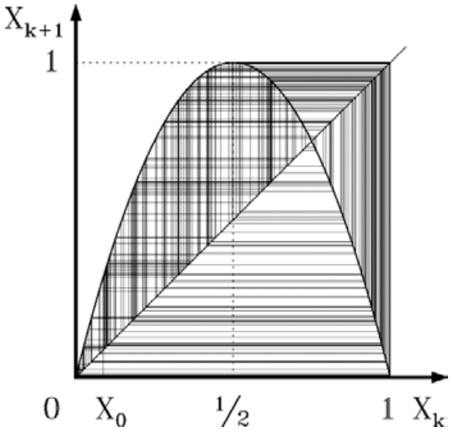

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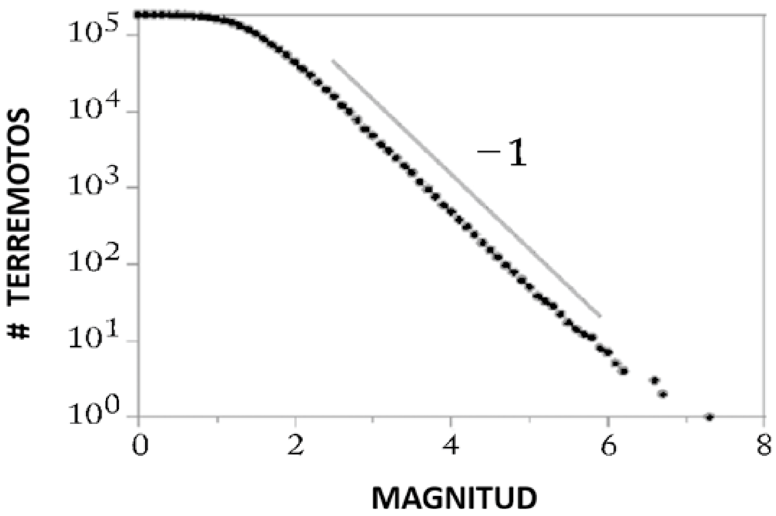
- This "**butterfly effect**" was first recognized while studying the *weather*.

Other power-laws

(Pareto, 1898; Schroeder, 1992; Turcotte, 1997)

Other power-laws

- They appear describing the frequency of complex natural events, such as ***earthquakes***, $P[X \geq x] \sim x^{-c}$, and they yield ***straight lines*** in ***double-logarithmic*** scales ($\ln P \sim -c \ln x$):



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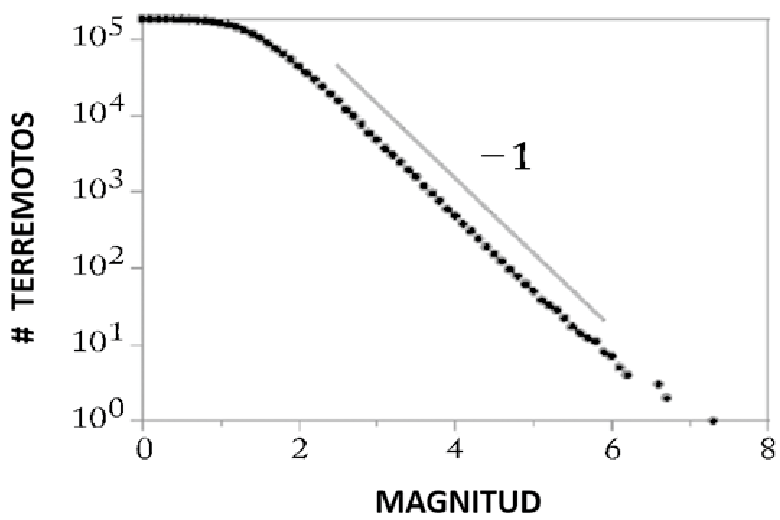

- These **Pareto curves**, with "***heavy tails***" and lacking ***characteristic scales***, appear in ***natural violence***, in ***avalanches***, ***forest fires***, etc., and also in the distributions of ***wealth*** and ***conflicts*** implicit in ***human fragmentation***.

…Well, here ends this brief introduction.

Next time we shall see how based on these notions it may be shown that Jesus is the way, the truth, and the life.

Until next time…

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