# THE ELOQUENCE OF TRANSFORMATION

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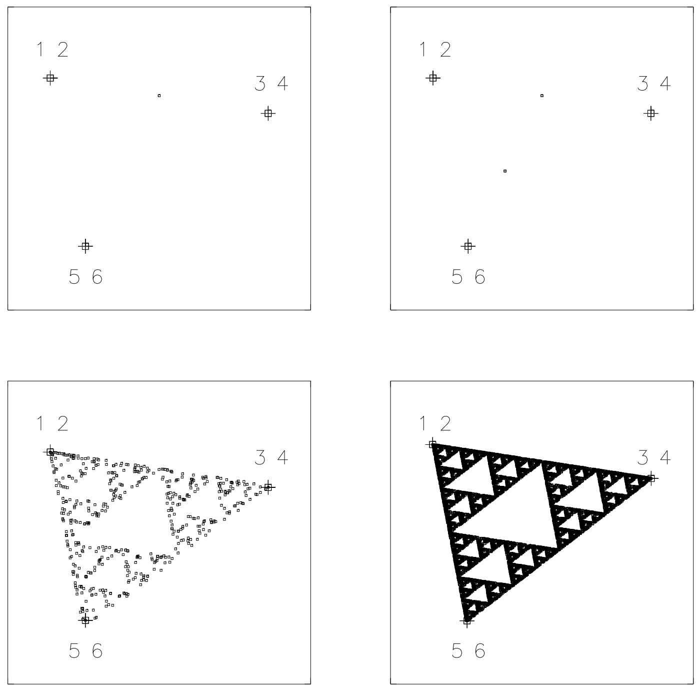
# Outline

- *Recalls the construction of the Sierpinski gasket via iterations of simple maps.*
- *Introduces other linear maps that generate "wires" shaped as clouds and mountains.*
- *Displays "shadows" of wires that provide a Platonic approach to natural complexity.*
- *Explains how the plane-filling wires universally yield bell curves as shadows.*
- *Generalizes the notions to higher dimensions including limiting bivariate bells.*
- *Exhibits unexpected and exotic kaleidoscopes of patterns decomposing circular bells.*
- *Shows ice crystals and biochemical rosettes as deterministic treasures inside the bell.*
- *Argues that "gentle freedom," in the "plenitude of love," is a great option.*

- This is a very special lesson for me, for the results herein summarize my own scientific work, as carried at Davis with the generous assistance of my collaborators.

- Recall the *Sierpinski gasket*, found *iterating* three simple maps that *"move to the middle"* of a vertex, as indicated by the *random* outcomes of rolling a die:




- By replacing such maps by other *inwardly-moving* rules, Michael Barnsley showed in 1988 that such generate other interesting **attractors**.

- Consider the two simple **linear** *"affine"* maps on the plane:

$$w_1(x,\ y)\ =\ (x/2,\ x + d_1 \cdot y)$$

$$w_2(x, y) = (x/2 + 1/2, 1 - x + d_2 \cdot y)$$

  with *vertical scaling parameters* $d_1$ and $d_2$ having magnitudes less than 1 and for $x \in [0, 1]$.

- The $x$ components depend only on $x$ and the $y$ components depend both on $x$ and $y$.

- On $x$, $w_1$ operates to the *left* of 1/2 and $w_2$ does so to the *right* of 1/2.

- The **attractor** generated by such maps may be glanced as follows.

- First of all, such includes the points (0, 0) and (1, 0), for $w_1$ and $w_2$ keep such points *"fixed:"* that is, $w_1(0,0) = (0,0)$ and $w_2(1,0) = (1,0)$.

- As $w_1(1,0) = (1/2,1)$ and $w_2(0,0) = (1/2,1)$, the attractor also includes the point $(1/2,1)$.

- The remainder of the attractor "springs" from images of $(1/2,1)$ via the two mappings:

$$w_1(1/2,1) = (1/4,\ 1/2 + d_1),\ \ w_2(1/2,\ 1) = (3/4,\ 1/2 + d_2)$$

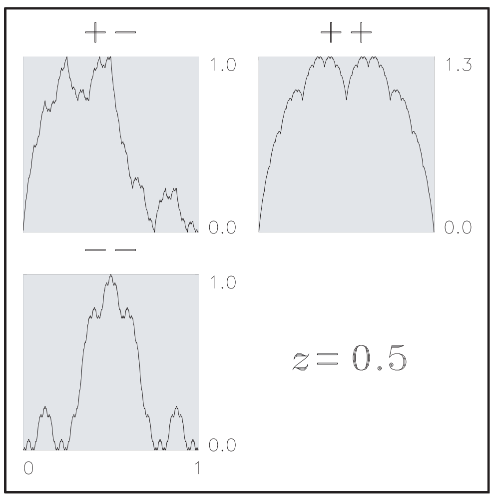
  which themselves yield new **intermediate** points, as in an *infinite binary tree*, e.g.,

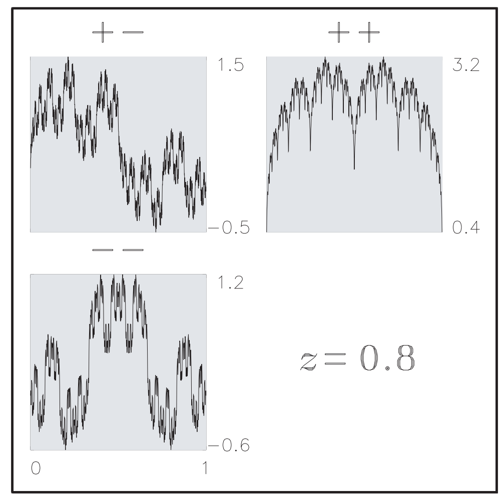
  $w_1(1/4,\ 1/2{+}d_1) = (1/8,\ 1/4{+}d_1{\cdot}(1/2{+}d_1)),\ w_1(3/4,\ 1/2{+}d_2) = (3/8,\ 3/4{+}d_1{\cdot}(1/2{+}d_2))$

  and so on for $w_2$, and ad infinitum.

- It turns out that by playing the *chaos game* with the simple maps, that is, iterating them via random tossings of a coin, yields a family of continuous "**wires**" that *interpolate* the aforementioned three points $\{(0,0),(1/2,1),(1,0)\}$. (!)

- Their graphs for $z = |d_1| = |d_2|$ and for all sign cases are: (the $-+$ wire is $+-$ flipped)

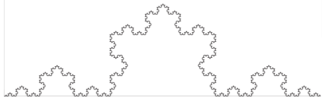


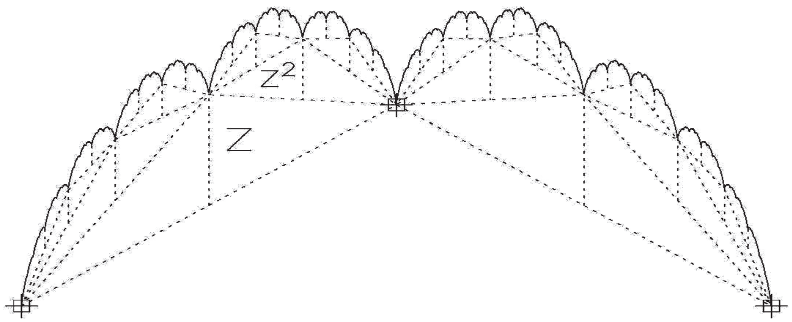

- Clearly, the wires on the left require less ink than those on the right.

- While the sets on the left are *finite* in length and not fractal, i.e., $D = 1$, those on the right are *infinite* in length and *fractal*, with dimension $D = 1 + ln(2 \cdot z)/ln(2) \approx 1.68$.

- As $z$ tends to 1, $D$ tends to 2, and hence the wires "*keep on growing*" to **fill-up** the plane.

- These sets may also be described geometrically as the *Koch curve*: 

- Following the math of the maps, the $++$ case, yielding a "**cloud**" profile, is as follows:

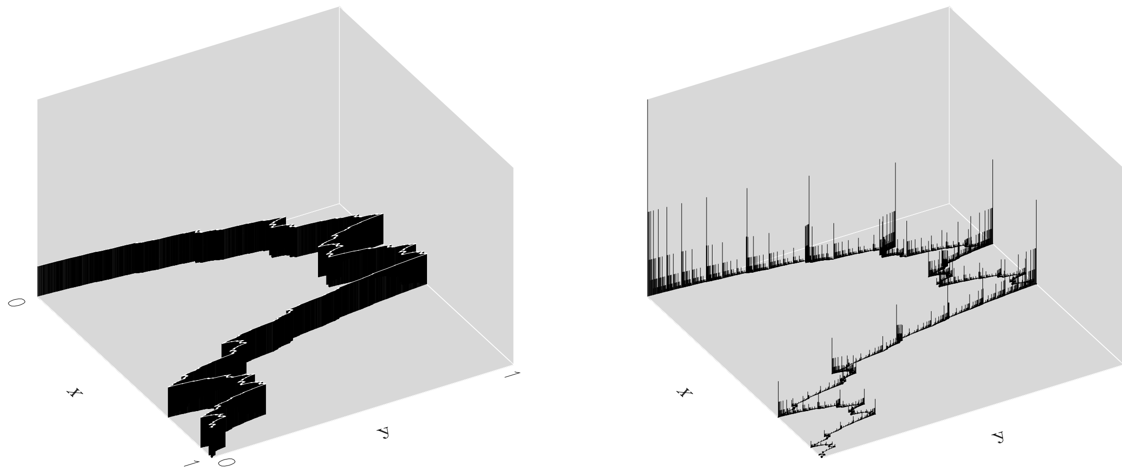



  - Join the three initial points from left to right.
  - Find two new points going **up** $z$ from the middle: i.e., $1/2 + d_1$ and $1/2 + d_2$ as before.
  - Join left to right to get four intermediate points **up** $z^2$, and so on, in powers of $z$. (!)

- For the cases $+-$ and $--$, yielding "**mountain**" profiles, there are similar constructions:

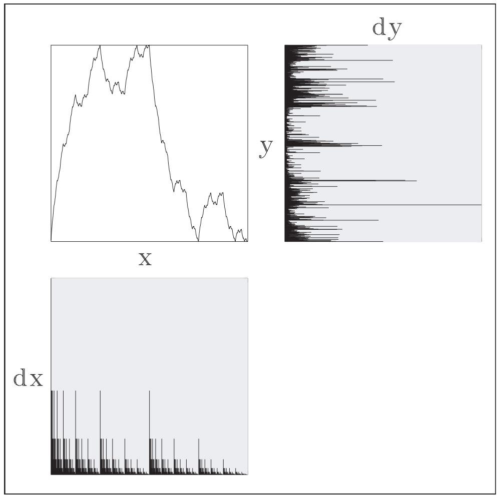
  - The $+-$ combination goes **up** and **down** on the first generation, then follows sequences of ups and downs on every step.
  - The case $--$ finds all its points alternating: the first two going **down**, the next four going **up**, and so on.

- The chaos game "paints" the wires *point by point*, and such appear irrespective of the type of "*coin*" guiding the iterations, due to an *ergodic* theorem proven by John Elton in 1987.
- However, and as illustrated below for the $+-$ case and $z = 1/2$, usage of **fair** or **biased** coins results in distinct stable *textures* defined over the wire:




- When the coin used is *fair*, the attractor is filled *evenly*, resulting in a *uniform* histogram.
- This is because maps $w_1$ and $w_2$ operate to the *left* and to the *right*, splitting the wire's domain precisely by the middle. (!)
- When the coin has a "70-30" *bias* relative to $w_1$ and $w_2$, the iterations generate a *multi-fractal* texture over the wire. (!)

- As from $x$ one gets the same *thorny* object of **turbulence** found via a *multiplicative cascade*, say $dx$, one good day it occurred to me to study the implied "*shadows*" of the texture defined over a wire, as seen from the $y$ axis, say $dy$.

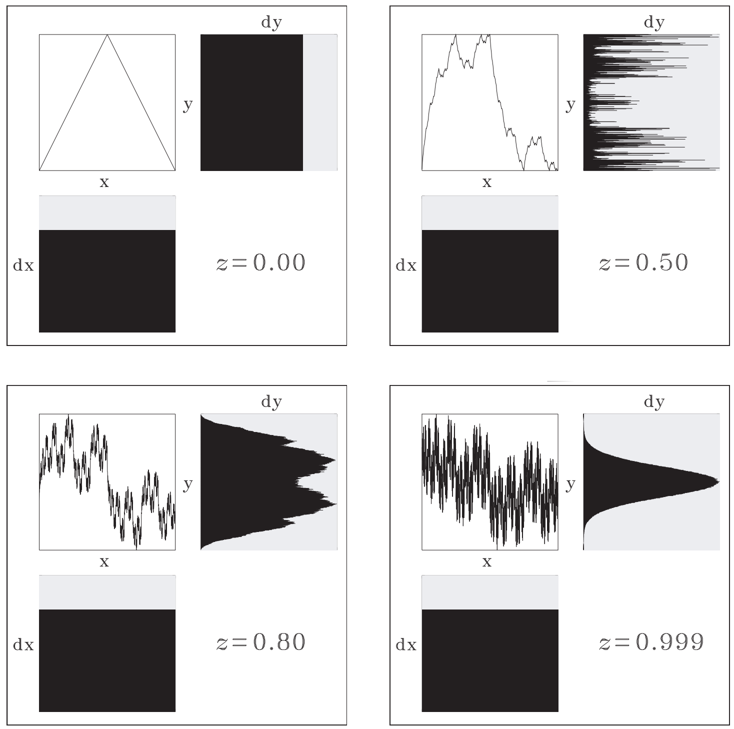



- As the wire is a *function* from $x$ to $y$, it also occurred to me that the idea could be useful to model hydrologic phenomena, for, after all, such processes often may be thought of as "*transformations of turbulence.*"

- The idea turns out to provide a "*Platonic approach to natural complexity.*" Platonic because of the celebrated allegory of the caveman and also because of the "romanticism" in suggesting an alternative model to the well-established notion of *randomness*. (!)

- A host of interesting patterns may be obtained by varying the implied parameters.

- As an example, as $z$ increases, one finds the following sequence from a *fair* coin:




- A *uniform* $dx$ yields *"projections"* $dy$ that count the *crossings* of wires by horizontal lines.

- As $z$ tends to 1, the wire fills-up the plane and one finds a **bell curve** as a "shadow," that is, the famous **Gaussian** or **normal** distribution, named after Karl Friedrich Gauss who used it first to study astronomic data in 1809. (!)

- The **bell curve** with *mean* $\mu$ and *variance* $\sigma^2$ is given by the familiar formula:

$$dy(y) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-(y-\mu)^2/2\sigma^2}$$

which contains the three key irrational numbers $\sqrt{2}$, $\pi$ and $e$.

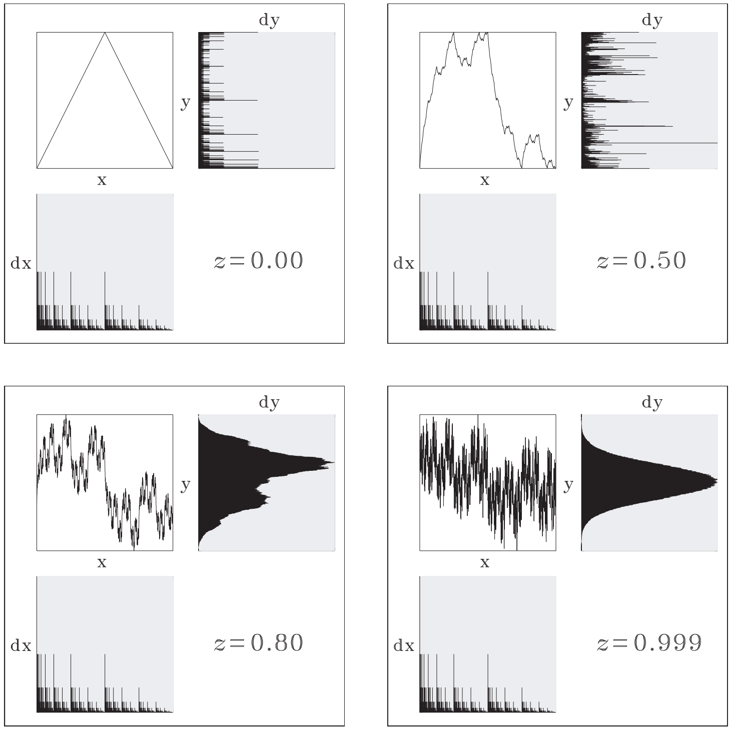
- Its relevance stems from its ubiquitous presence in science (e.g., *diffusion dynamics* and *heat conduction*) and its relation to the rather "mystical" *central limit theorem*, which asserts that sums of *independent* "random" events, under a host of conditions, give *bells*.

- As the sum of independent events *uniformly distributed* yields a *normal distribution*, the fact that the bell is found as the shadow of a plane-filling wire appears to be just a consequence of a central limit theorem.

- However, and as it shall be further appreciated later on, the **geometric construction** is indeed *new* and appears not to be implied by existing theorems.

- A proof of this surprising limiting result was advanced in 1996 under the title "The Gaussian distribution revisited," and such relies on showing *by induction*, via rather lengthy analytical expressions, that the moments of $dy$ indeed converge to the moments of the Gaussian distribution as $z$ tends to 1. (!)

- Similar to the previous sequence, one gets the following for a 70-30 *biased* coin:




- $dy$ is the "*derived distribution*" of $dx$ via the *transforming* wire and such are found, for a given height $y$, adding the "masses" in $x$ associated with such a value.

- As the wires become more intricate, they increasingly filter the spikes of the "illumination."

- In the limit, the **same wire** as shown before via a fair coin, yields a *bell curve* and such happens irrespective of the *multi-fractal input*, that is, it occurs for **any** value of $p$. (!)

- This result turns out be even stronger, for one may use as $dx$ any texture with a *continuous cumulative distribution* over the domain $[0, 1]$, and such would also lead to another bell, a bell whose *mean* and *variance* would depend on such a $dx$. (!)

- This includes, besides **smooth densities** over $[0, 1]$, any *spiky textures* over *uncountable dusts* as found before in these lectures and that lead to **devil's staircases.** (!)

- These latter results are unexpected, for unlike what happens to smooth densities, adding independent events whose distributions are *multi-fractal* yields yet more *multi-fractals*, i.e., for such textures do not belong to the "*domain of attraction of normality*," as their ever-present *thorns* and *dust* are not covered by the *central limit theorem*.

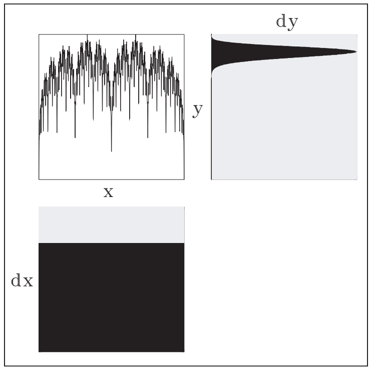
- The limiting space-filling wire is certainly rather special, for **only** such provides, in its inherent and maximal infinitude, a **universal** *transformation* from "**disorder**" into "**harmony**" and an unforseen *bridge* from **turbulence** to **diffusion**. (!)

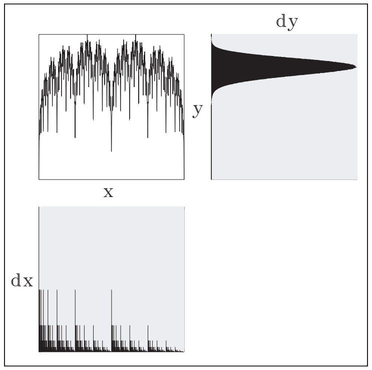
- The only cases excluded from normality are **discrete** textures defined over *finite* or *countable* points, which surely give a shadow that can not coalesce into a bell over a continuum.

- As all of these is found for the $+-$ case, one wonders if bells also happen for the other sign combinations.

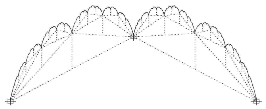
- The *mountain wire* of the $--$ case turns out to give not *one* but *two* bells. This happens as the defining **downs** and **ups** lock, in the limit, into two oscillating means. (!)

- The *cloud wire* corresponding to the $++$ case ends up being rather special. When both $d_1$ and $d_2$ tend to **one**, one finds, *universally*, a bell **concentrated** at *infinity*: (!)






- As hinted from the above graphs for $z = 0.99$, both the mean and the variance tend to infinity, but they do so in such a way that $\sigma/\mu \to 0$. This means that, ultimately, "*all the relevant mass,*" with probability one, is *lifted up* into *infinity* in a manner that annuls all *entropy* present in any *non-discrete* input $dx$. (!)



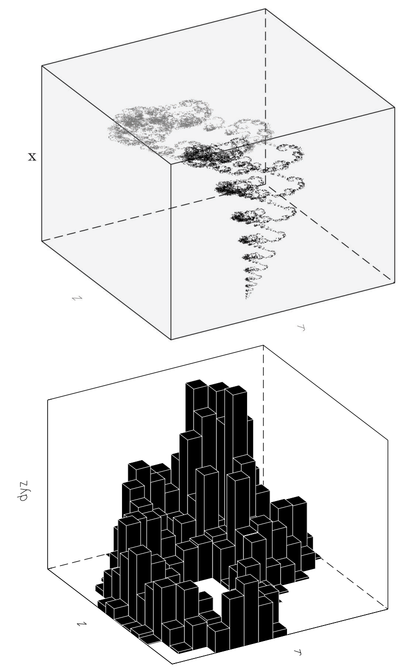
- Quite beautifully, this **mighty transformation**, shaped as "*angel wings,*" takes it all to the "clouds," and such happens following the melodious moments of the bell. (!)

- The ideas may be extended to *higher dimensions* adding more *coordinates* and *parameters*:

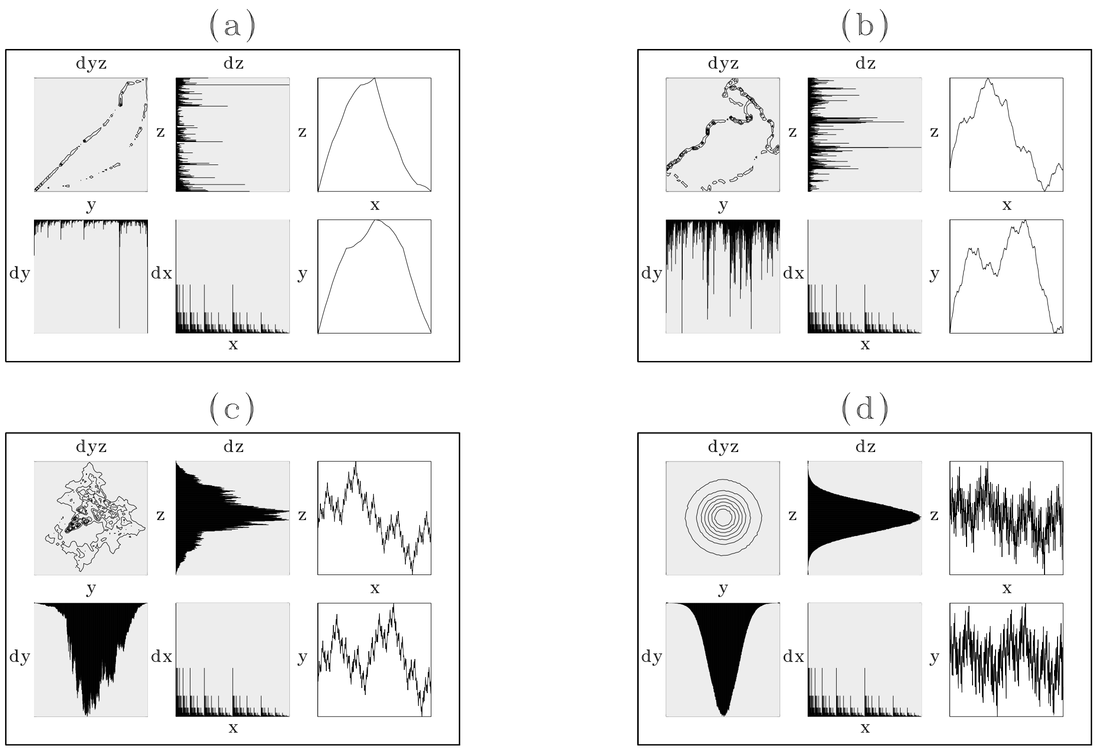
$$
\begin{aligned}
w_1(x,y,z) &= (x/2, Hx + d_1 y + h_1 z, x + l_1 y + m_1 z) \\
w_2(x,y,z) &= (x/2 + 1/2, -Hx + d_2 y + h_2 z + H, -x + l_2 y + m_2 z + 1)
\end{aligned}
$$

- Such simple maps yield *"fractal" wires* in three dimensions, from $x$ to the plane $y - z$, that interpolate $\{(0{,}0,0),(1/2{,}H,1),(1{,}0{,}0)\}$ and have *dimensions* between **one** and **three**.

- With such new wires one may obtain projections over the $y - z$ plane, to generalize the *Platonic* notions to *natural complexity*:



- *Space-filling wires* yield **bells**, now two-dimensional, $dyz$, and they appear based on any *multi-fractal* $dx$, (for $d_1 = d_2 = m_1 = m_2 = 0$, $h_1 = -h_2 = l_1 = l_2 = z$, $H = 1$): (!)



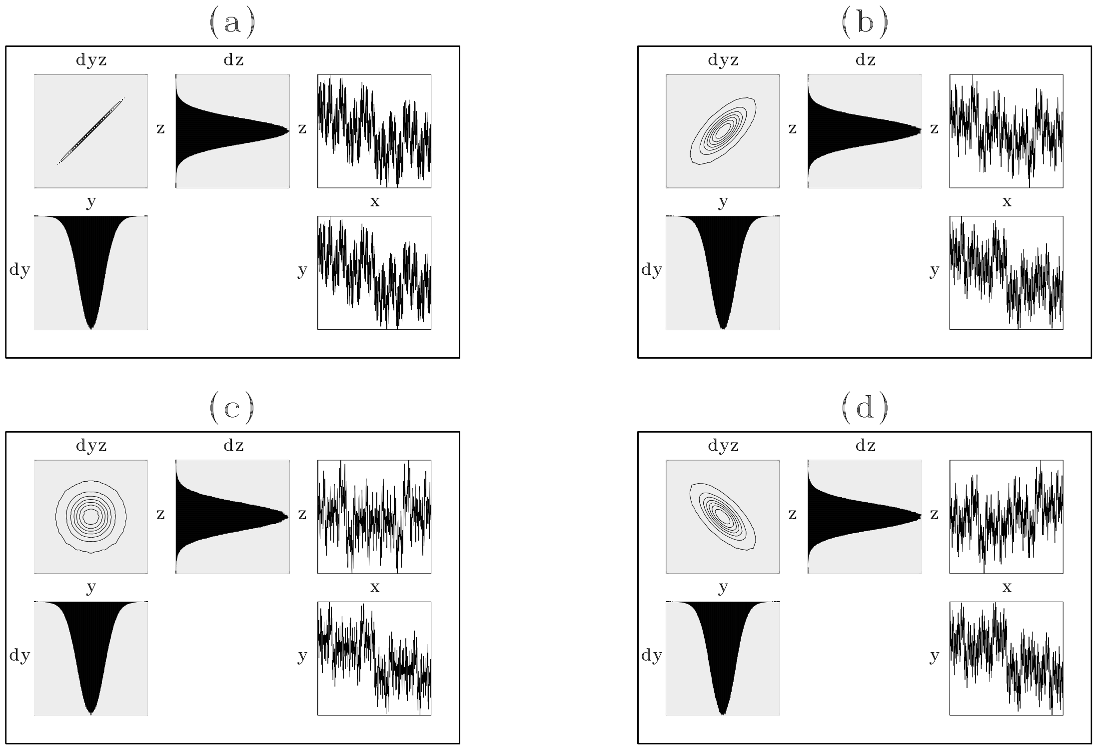

  - (a) $z = 0.25$, (b) $z = 0.5$, (c) $z = 0.75$, (d) $z = 0.999$.

- It is convenient to use *"polar coordinates"* to have *scalings* and *rotations* as parameters:

$$\begin{pmatrix} d_i & h_i \\ l_i & m_i \end{pmatrix} = \begin{pmatrix} r_i^{(1)} \cdot \cos\theta_i^{(1)} & -r_i^{(2)} \cdot \sin\theta_i^{(2)} \\ r_i^{(1)} \cdot \sin\theta_i^{(1)} & r_i^{(2)} \cdot \cos\theta_i^{(2)} \end{pmatrix}, i = 1, 2.$$

- A **bell** is found when the magnitude of all *scalings* $|r_i^{(j)}|$ tend to 1 and when the *angles* on every map are synchronized such that $\theta_i^{(1)} = \theta_i^{(2)} + k_i\pi$, for an integer $k_i$.

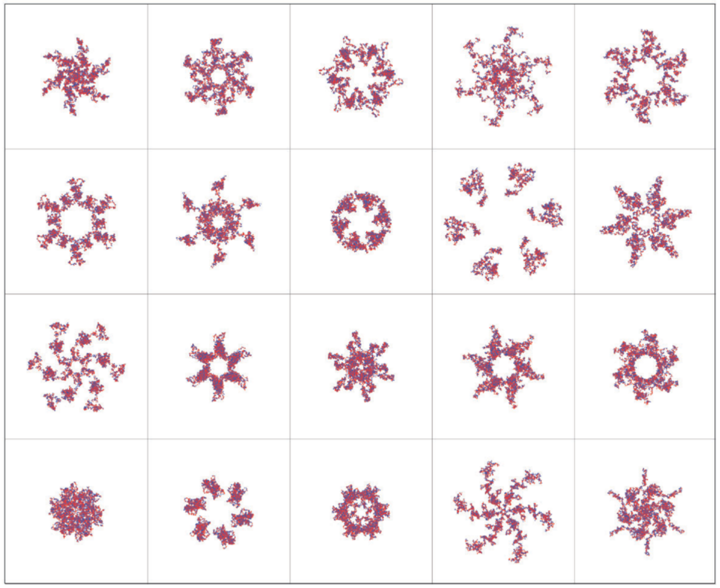
- Elliptical bells having arbitrary *correlations* may be found from a limiting wire and a *uniform* texture (for ${r_1}^{(1)} = -{r_1}^{(2)} = -{r_2}^{(1)} = {r_2}^{(2)} = 0.999$, ${\theta_1}^{(1)} = {\theta_1}^{(2)} = {\theta_2}^{(1)} = {\theta_2}^{(2)} = \pi/2$): (!)




  - (a) $H = 1$, (b) $H = 0.41$, (c) $H = 0$, (d) $H = -0.41$, with *coefficients of correlation*, $\rho$, of, in order, 1, 0.7, 0, and -0.7.

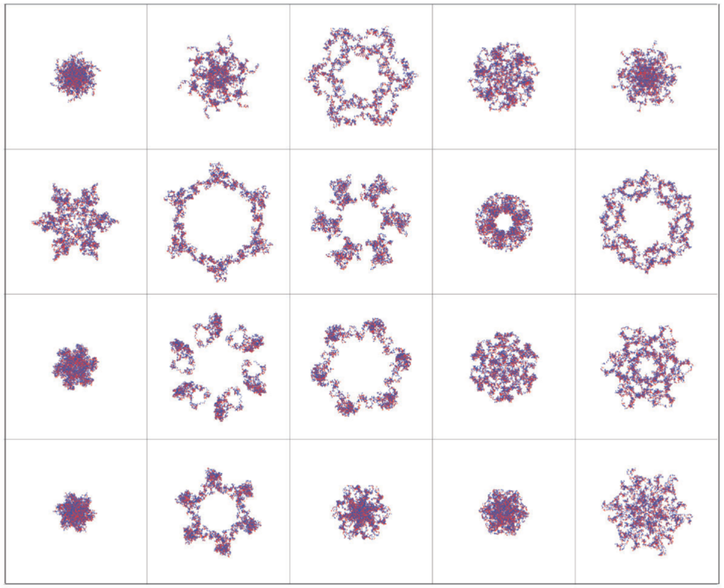
- As the previous diagrams summarize what is found via many iterations, e.g., 15 million, and as a proof for the higher dimensional bells eluded us, on another good day it occur to us to study how bells are formed when considering few thousand points at a time.

- If the *angles* are further *"synchronized,"* i.e., $\theta_1 = {\theta_1}^{(1)} = {\theta_1}^{(2)}$, $\theta_2 = {\theta_2}^{(1)} = {\theta_2}^{(2)}$ and both $\theta_1$ and $\theta_2$ divide $2\pi$ and are multiples of one another, *circular bells* are decomposed in terms of **kaleidoscopes** (for a **fair** coin, every 10,000 dots colored *red* and *blue*, top-bottom and left-right, $H = 1, {r_1}^{(1)} = {r_1}^{(2)} = {r_2}^{(1)} = {r_2}^{(2)} = -0.9999, \theta_1 = 2\pi/3, \theta_2 = \pi/3$): (!)



- Surprisingly, circular bells are not made of circles and disks, but of beautiful *"mandalas."*

- By using a **biased** coin one also gets lovely symmetric "*treasures*" (with the same parameters as before and for a 30-70 coin yielding "bluer" patterns): (!)



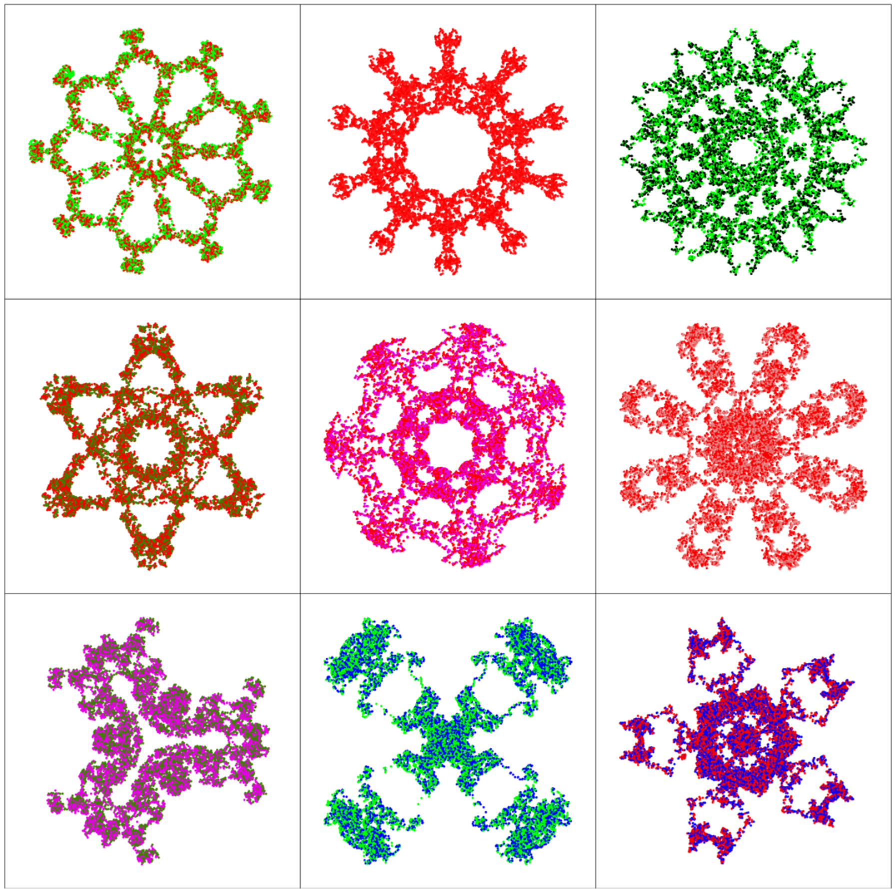
- Holding the scalings at "*many nines*" and by varying the angles, one finds beautiful sets with arbitrary $n$-fold *symmetries*. (!)

- The actual patterns are **infinite** and they depend strongly on the chosen iteration pathway, that is, on the very precise sequence of "coin tosses" used.

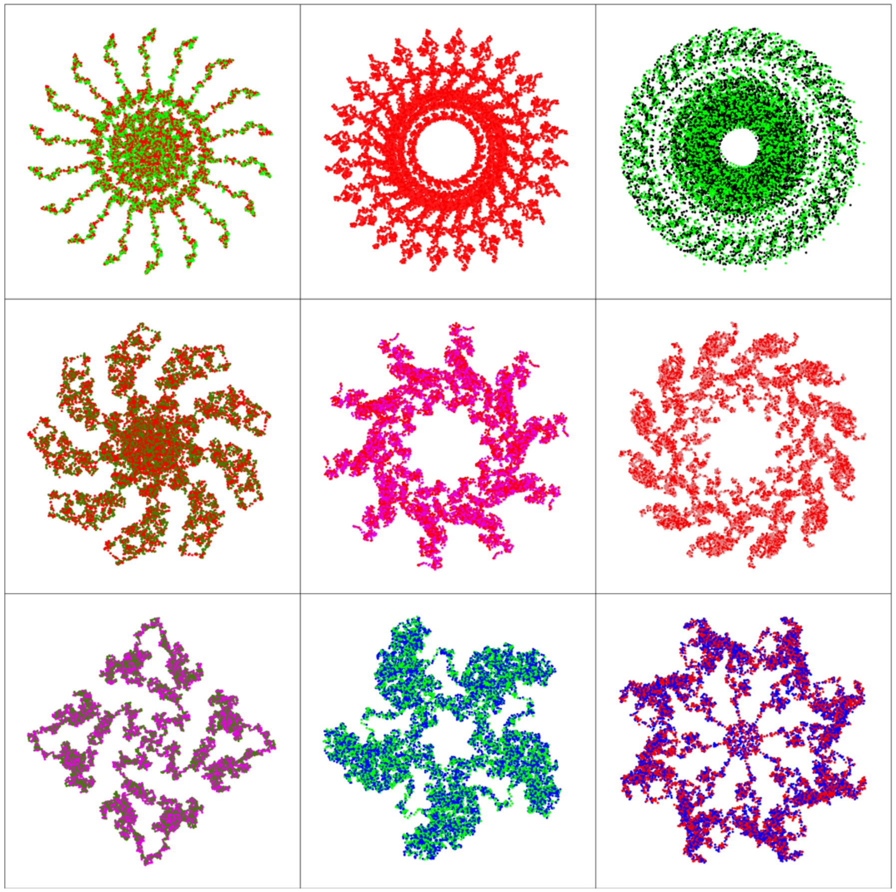
- The emergence of pattern after pattern after pattern remind us of the story "*The Aleph,*" written by the Argentine philosopher and writer Jorge Luis Borges in 1949. (!)

- Depending on the sign combinations of the *scalings* $r_i^{(j)}$, the resulting patterns have either *radial* or *rotational* traits.

- A sample of the "*radial*" mandalas (for *fair* coins and 20,000 dots, varying **angles** and $H = 1, r_1^{(1)} = r_1^{(2)} = r_2^{(1)} = -r_2^{(2)} = 0.99999999$) are:

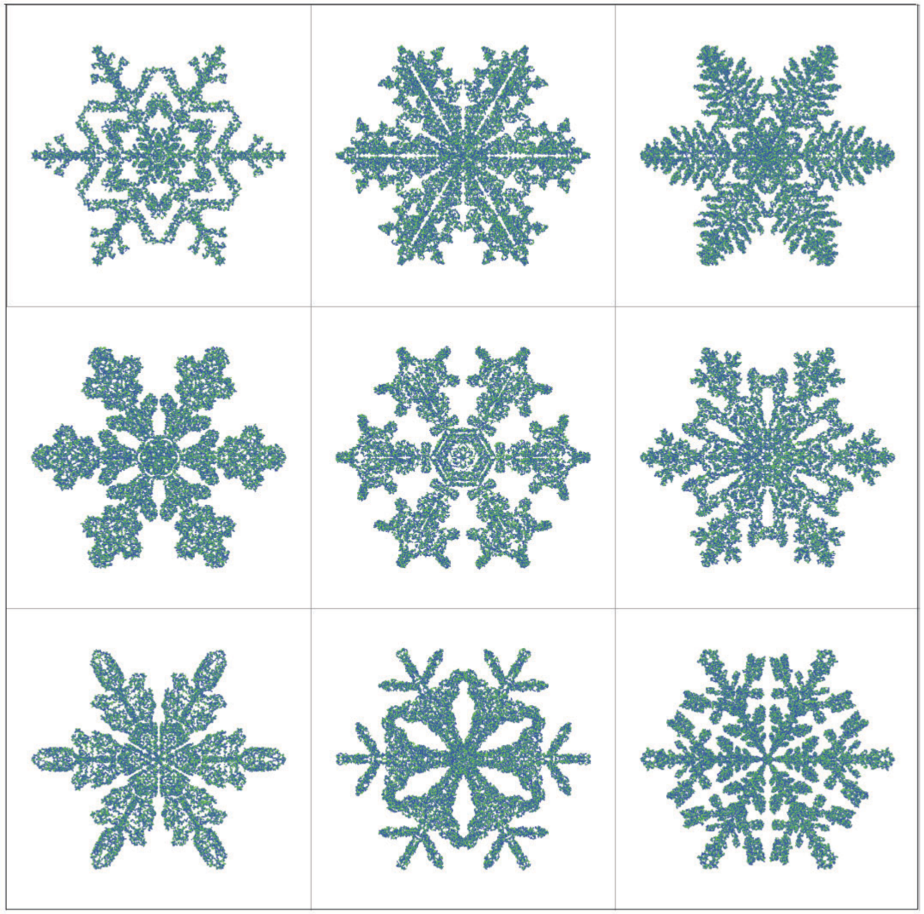


- The patterns have exquisite dynamics and, if the iterations are prolonged, points would dance in and out eventually making up the bell. (!)
- Some examples of "*rotational*" mandalas (for *fair* coins and 20,000 dots, varying **angles** and $H = 1, r_1^{(1)} = r_1^{(2)} = r_2^{(1)} = r_2^{(2)} = 0.99999999$) are:



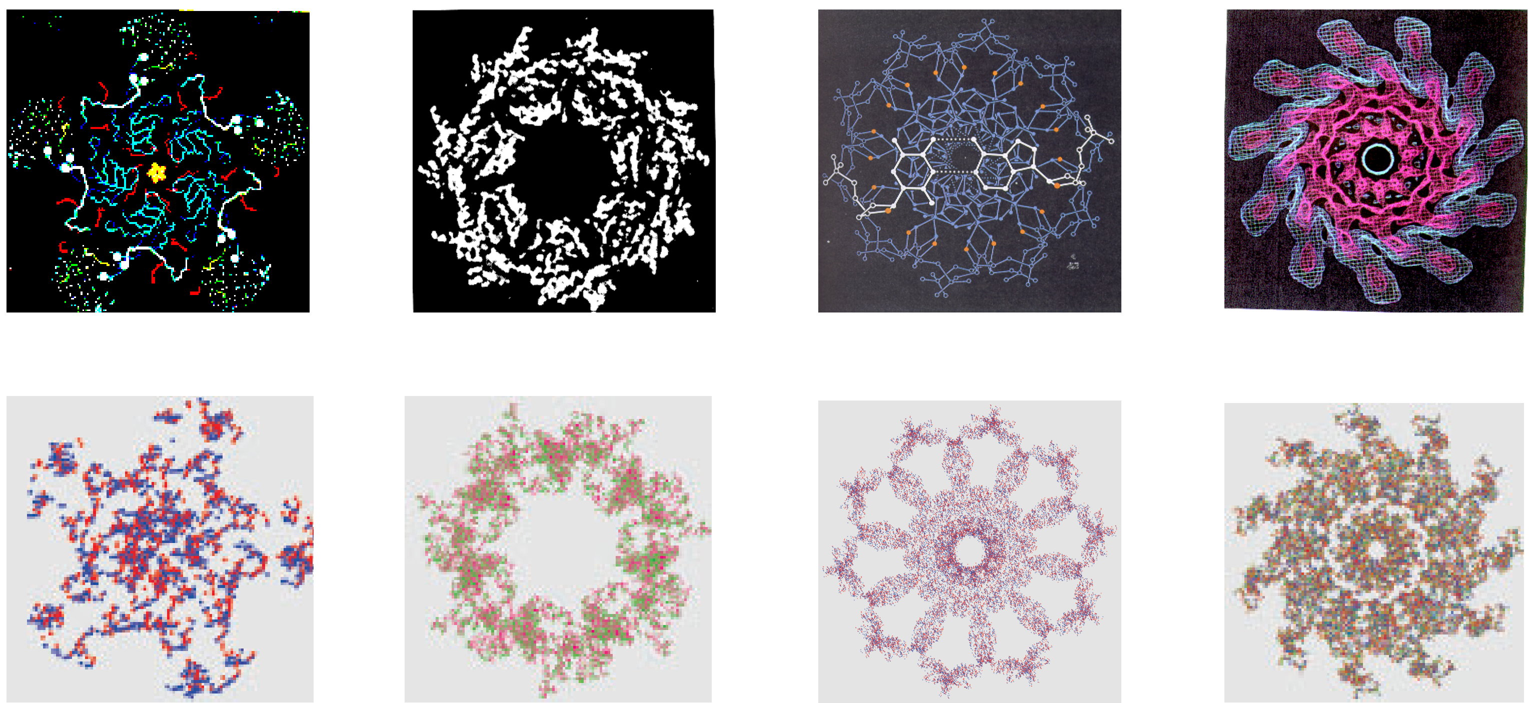
- Some patterns inside the bell turn out to resemble relevant natural sets.

- It is easy to "*grow*" **ice crystals** inside the bell (for suitable sequences of iterations, *angles* $\theta_1$ and $\theta_2$ set at 60 degrees, and $H = 1, -r_1{}^{(1)} = r_1{}^{(2)} = r_2{}^{(1)} = r_2{}^{(2)} = 0.99999$): (!)



  – Templates of these sets from Bentley's catalog were filled by patient trial and error.

- There are also **biochemical rosettes** inside the bell. (!)
- They include, from left to right, the *foot and mouth virus*, the *E.coli GroEl protein*, the *B-DNA rosette*, and the *salmonella bacteria*, with the "real" patterns above and those inside the bell below:



  - Here only the pattern of DNA was "optimized," selecting a pathway of iterations that fills a template of such a rosette.

- As the shown "**treasures inside the bell**" are just **deterministic designs**, inside the bell there is "**hidden order in chance.**" (!)


- These beautiful results suggest a *central* role for the Gaussian distribution beyond the familiar fields of physics, probability, and statistics, that also encompasses biology.

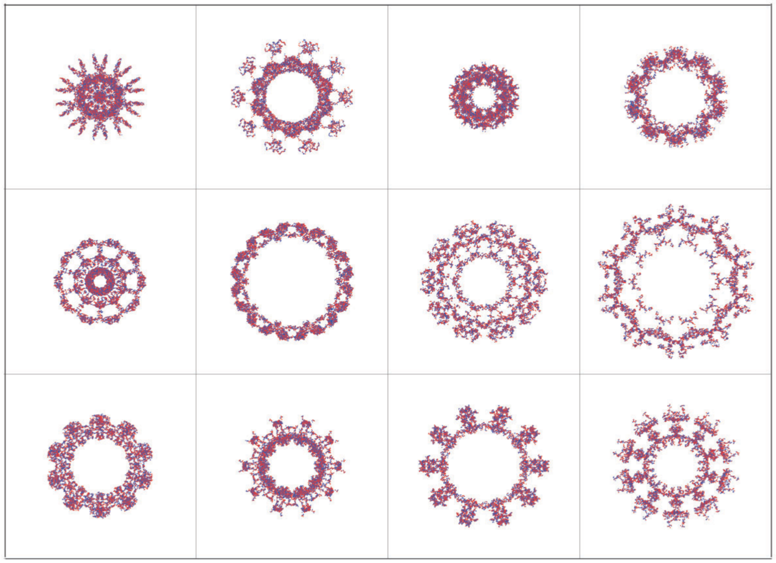
- As there is always **beauty in the limit** and the kaleidoscopes emerge only when a wire *fills-up space*, the results provide an alternative *paradigm* for the *origins of order*, one that may be coined "**order at the plenitude of dimension**." (!)

- As one is "wired" to the ideas, one realizes that the framework yields a rather *compact* and rather *simple* algorithm for encoding *information*.

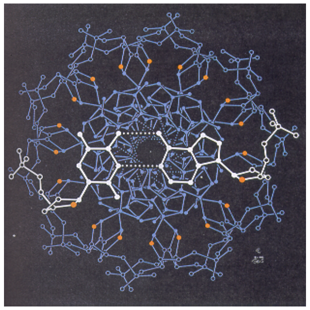
- For, just a single *infinite* wire, sampled via distinct pathways of iterations, surely yields a host of interesting patterns. (!)

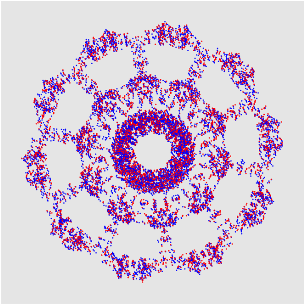
- In such spirit, and as the three irrational numbers $\sqrt{2}$, $\pi$, and $e$ are prominent in the formula of the bell, on yet another good day it occurred to me to study the lovely patterns surely hidden forever inside circular bells, when binary expansions of such numbers aid the iterations, i.e., $\sqrt{2} = 1.0110100\cdots$, $e = 10.101101\cdots$, $\pi = 11.001001\cdots$.

- The following patterns are found via the binary expansion of $\pi$ (for 20,000 dots, *angles* $\theta_1 = \theta_2 = 36$ degrees, and $H = 1, -r_1^{(1)} = r_1^{(2)} = r_2^{(1)} = r_2^{(2)} = 0.99999999$):



- The second set provides a close rendering of the B-DNA rosette: (!)





- This curious result is certainly rather suggestive and surely "rings few bells." For at least it hints that the "holder of the celestial screwdriver" may be not just a *"blind watchmaker."*

- At this time we may pause a bit and reflect on the very rich symbolism present in this lecture.

- As before, I think that these ideas are useful to illuminate our personal choices.

- As some fractal wires *transform* patterns of *disorder* into other patterns of *disorder*, and other *special space-filling* wires *transform* patterns of *disorder* into patterns of *order*, I sense that the notions herein help us visualize our "**spiritual**" options and vividly remind us of the possibility of "**transformation**."

- As the $++$ bell at *infinity* keeps "*the flame*" and contains *no entropy*, the ideas reinforce that it is best, if at all possible, for us to **conduct** the *joy of life* rather than to *dissipate* it with our *negativism*. (!)

- For there is a marked difference between *finiteness* and *infiniteness*, even for the $+-$ bell, and such distinction help us appreciate why seeking "**plenitude**" in the "*highest possible dimension*" of **Love** may be better than to continue living in inherent *solitude* and *fear*, in a condition that, quite accurately from fractal dimensions, can never "*fill us up.*" (!)

- For we may indeed choose to *trust* and go forward saying "**yes** to **God**," that is and as explained in previous lectures, practicing the outward spiral $0.999\cdots$ that brings *unity*, or we may *rebel* to attempt to live "*our way*," essentially negating the power of transformation.

- This translates into choosing "**faith**" as our transforming motto or clinging to *doubts* that restrict our growth. For we could choose the *gentle freedom* symbolized by the ever gracious bell or continue being *slaves* to our *ups* and *downs*. (!)

- In the spirit of the *central limit theorem*, we may surrender to live *smoothly* and **independently**, "*picking up our positive cross one day at a time*," as the Bible says, or we may we may live in sub-optimal and all too-common states characterized by "*thorns*" of *anxiety*.

- We may choose to exalt the *exotic beauty* concealed in ultimate bells or continue "admiring" *stagnant shades*, just patterns of disorder, at lower "spiritual" dimensions.

- At the end, we may appreciate our destiny here on earth, for, as explained in the Bible, we may choose between *light* and *darkness* and between *heaven* and *earth*.

- For, by exercising our "free will" we may accept the **plus**, the reality of **Love**, $I = +\sqrt{+1}$, or the *minus* that leads to an "imaginary" trap, $i = \sqrt{-1}$. (!)

- In conclusion, these are some of the options that this lecture reminds us of:

| | |
|---|---|
| *Conduction* | *Dissipation* |
| *Plenitude* | *Solitude* |
| *Trust* | *Rebellion* |
| *Faith* | *Doubts* |
| *Freedom* | *Slavery* |
| *A day at a time* | *Anxiety* |
| *Exotic beauty* | *Stagnant shades* |
| *Infinity* | *Finiteness* |
| *Light* | *Darkness* |
| *Heaven* | *Earth* |
| + | − |
| *Love* | *Else* |

- The following poem-song summarizes this lesson.

## THE AMAZING BELL

*By the mystery of science*
*graciously shines a state,*
*an all-embracing alliance*
*adding liberty a shape.*

*One day, as if by chance,*
*boldly there was such gem,*
*as the shadow off a wire*
*that fills completely space.*

*As the ideas hint above*
*enduring a lasting zest,*
*here is probable code*
*in the ever precious bell.*

The bell peals silent, oh oh
reflecting its peace,
and inside it gathers
lovely masterpiece.

Symmetric pure beauty, oh oh
oh mighty delight,
this limit in fullness
stores life's designs.

Such vessel contains, oh oh
alephs of all tastes,
diatoms and crystals
including DNA.

But there is a case, oh oh
reason to this song:
the forward selection
that raises it all.

There is clear choice
that rotates the 8.

**By loving sincerely**
**we surely converge.**

Notice, this is cogent:
the bell's central theme.

**By living in freedom**
**one fulfills the dream.**

There is transformation
that kindles the heart.

**By loving in plenitude**
**we become smart.**

For love mends the spiky
and takes to the clouds.

**By living the present**
**one joins blessed crowd.**

Oh see, this is truthful:
the plus all the way.

**By loving the enemy**
**we learn how to play.**

Dimensional growth,
oh essence of life.

**By living in harmony**
**one nails normal plan.**

Oh notice the symbols,
oh irrational might.

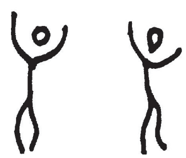
**By loving simplicity**
**we experience the light.**

Oh listen, you colleague,
let's go out the cave.

**By living in unity**
**we shall all prevail.**

Oh notice, my friend,
the plea from a bell.

**By loving and loving**
**joy will have no end...**



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