THE ESSENCE OF CHAOS

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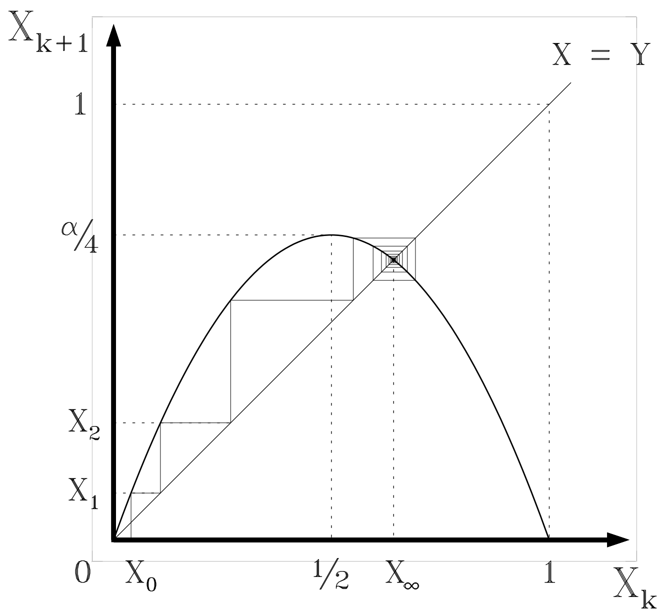
Outline

- *Introduces the logistic map and its incredible dynamics.*
- *Explains how such a map leads to alternative behaviors including periodicity and chaos.*
- *Exhibits the Feigenbaum tree and its amazing universal properties.*
- *Shows intertwined pathways leading to distinct destinations in the plenitude of chaos.*
- *Illustrates that there is an improbable way out of utter chaos.*
- *Argues from chaos theory that "abandonment," as defined in the Bible, is essential to enter the "Kingdom of God."*

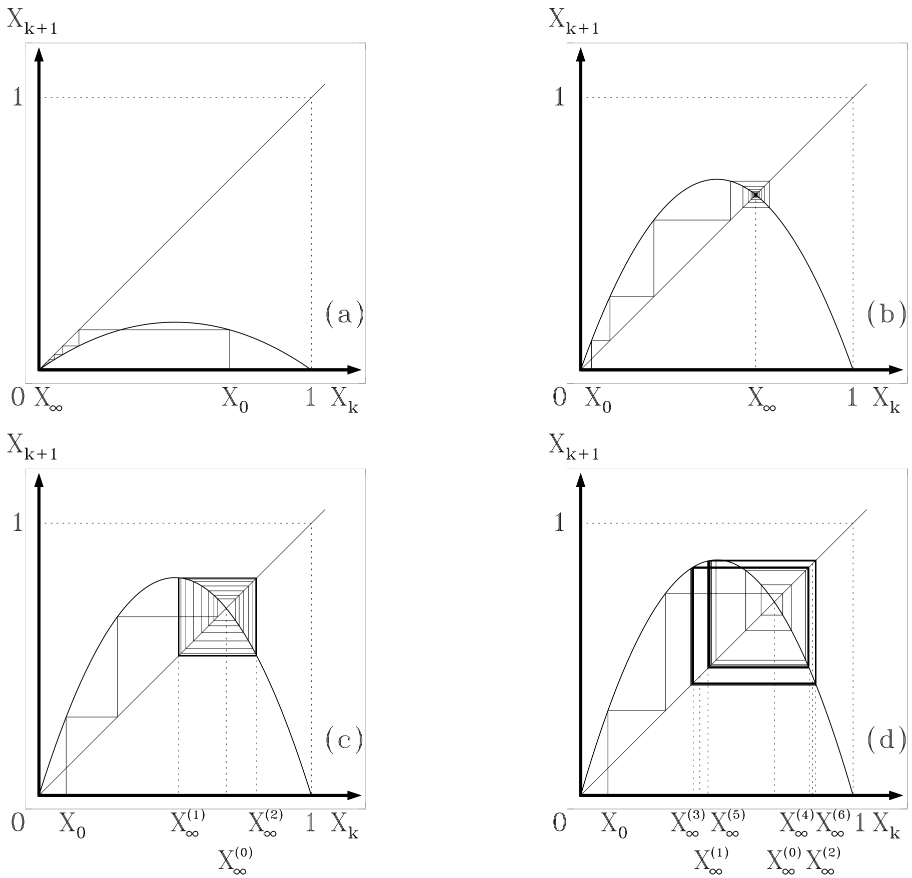
- Hailed as one of the most important scientific achievements of the 20th century, chaos theory turns out to provide useful and poignant symbols for the attainment of peace.

- The prototypical equation used to illustrate the well-established theory is the *logistic map*: $X_{k+1} = \alpha X_k(1 - X_k)$, $\alpha \in [0, 4]$. X is the size of a normalized "*population*" from 0 to 1, say rabbits, k and $k+1$ are subsequent generations, and α is a *parameter*.

- The logistic *parabola*, exhibiting an increase from generation to generation if the population is small but a decrease if it is large, is shown below ($\alpha = 2.8$) together with a sequence of *iterations*, an *orbit*, that leads to the *ultimate fate* of the population, X_∞.

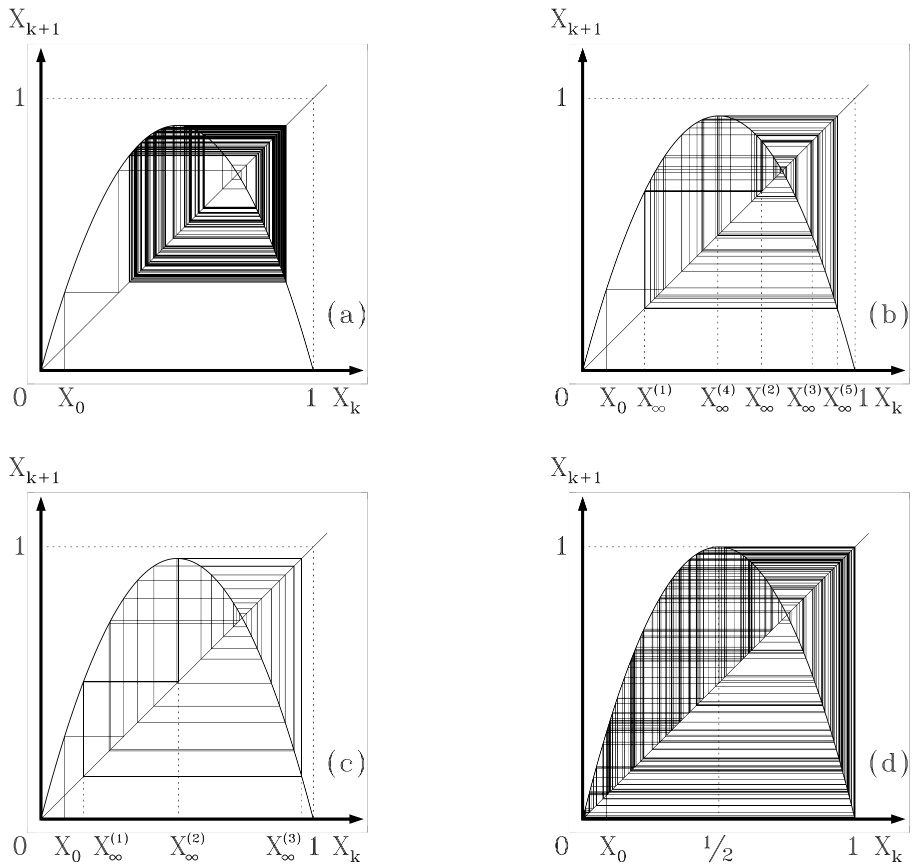


- The ultimate X_∞ happens to depend on the choice of α:



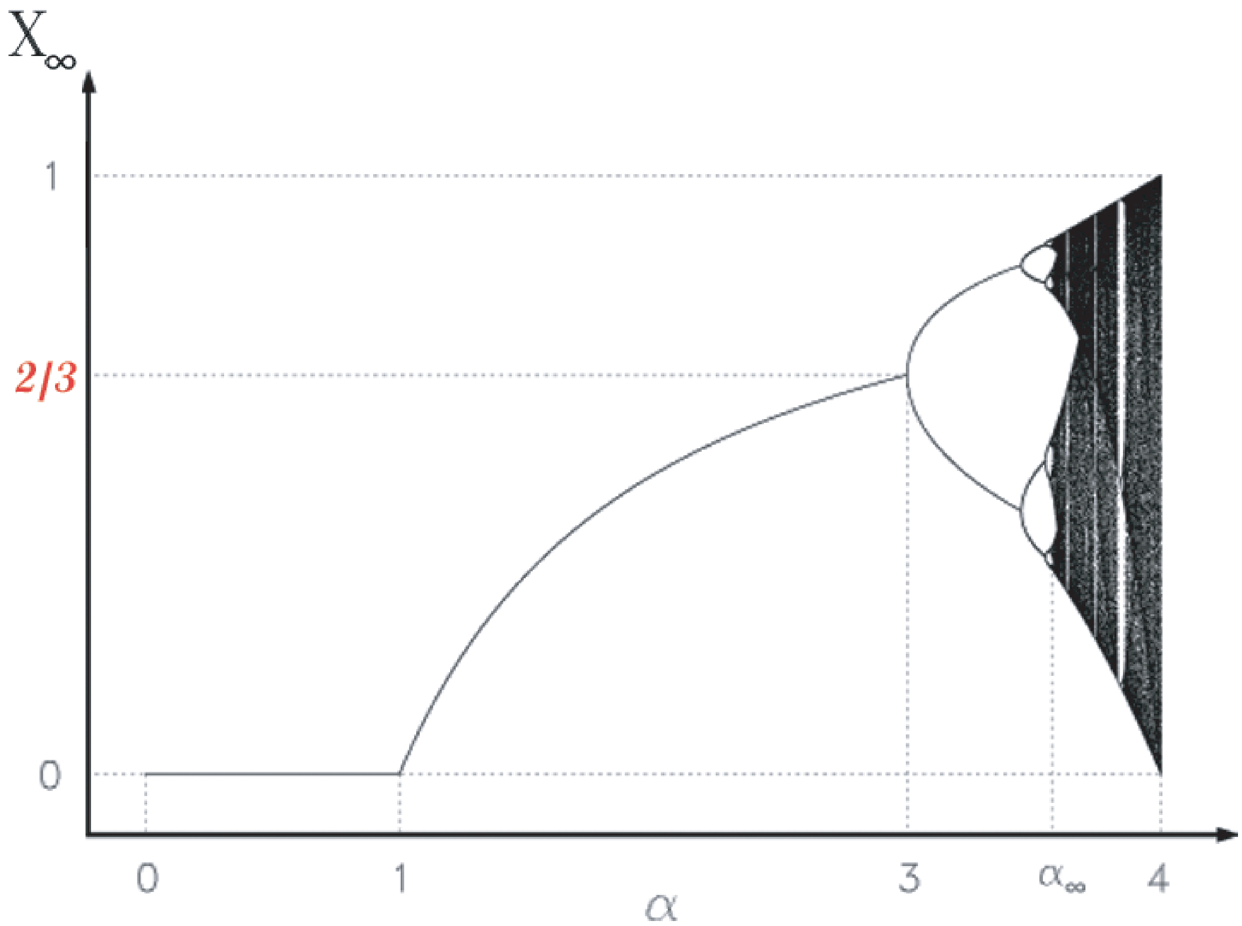
- If the parabola is *below* $X = Y$ as in (a), $\alpha = 0.7$, $X_\infty = 0$. *Zero* is a *stable attractor*.
- If parabola is *above* and $\alpha < 3$ as in (b), $X_\infty = (\alpha - 1)/\alpha$. X_∞ is now a *fixed point* attractor.
- If $\alpha > 3$ as in (c,d)($\alpha = 3.2, 3.46$), *oscillations* set in, every 2 or 4 generations. (!)
- When the parabola is above the line, **the origin** *repels*, as the slope there is too steep.

- Successive *bifurcations* occur for increasing values of $\alpha \leq \alpha_\infty \approx 3.5699$: that is, oscillations every 2 generations, every 4, every 8, every 16, and, quickly, every *power of 2*. (!)
- After α_∞ behavior is either *periodic* or *non-repetitive*:

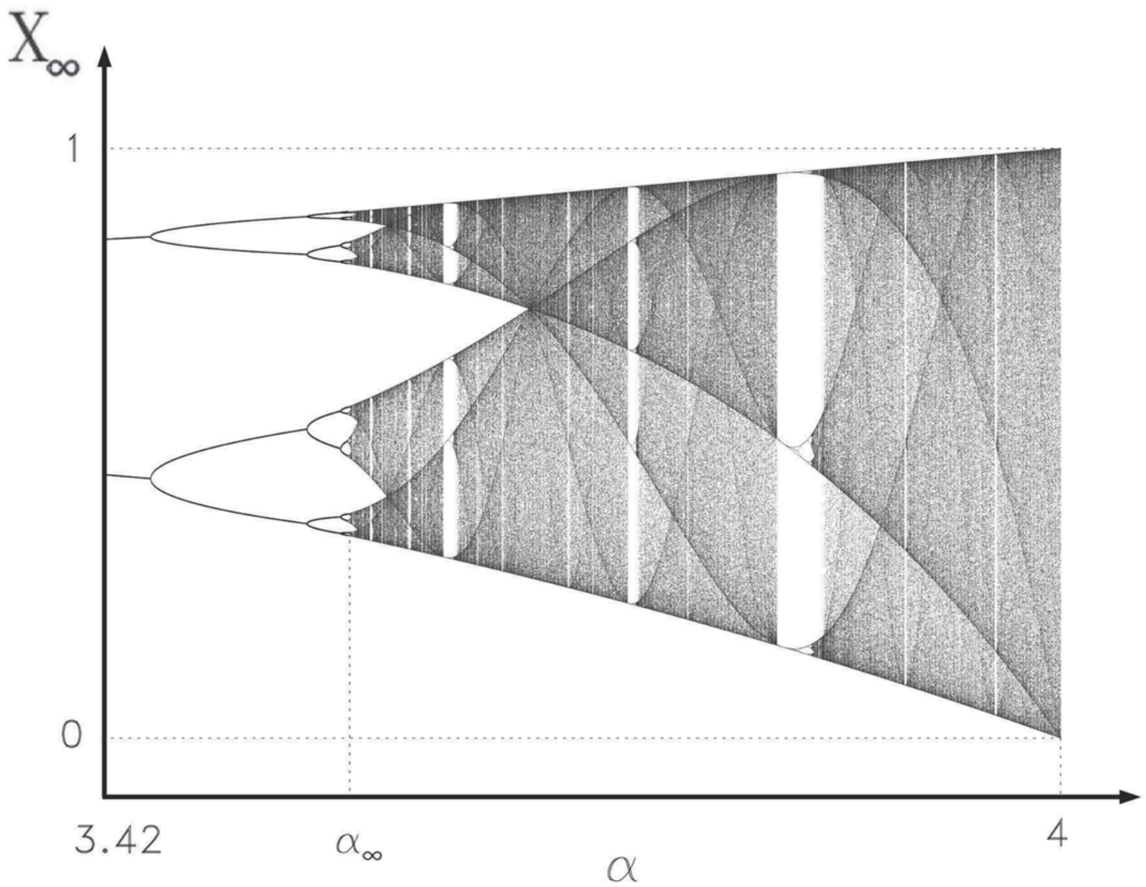


- Cases (a,d) ($\alpha = 3.6$, 4) yield infinite and non-countable **chaotic** "*strange attractors.*" (!)
- Cases (b,c) ($\alpha = 3.74$,3.83) give oscillations repeating every 5 and 3 generations. (!)

- The *stable attractor* of the dynamics, X_∞ as a function of α, is known in the literature as the *bifurcations diagram* or the *Feigenbaum tree*:



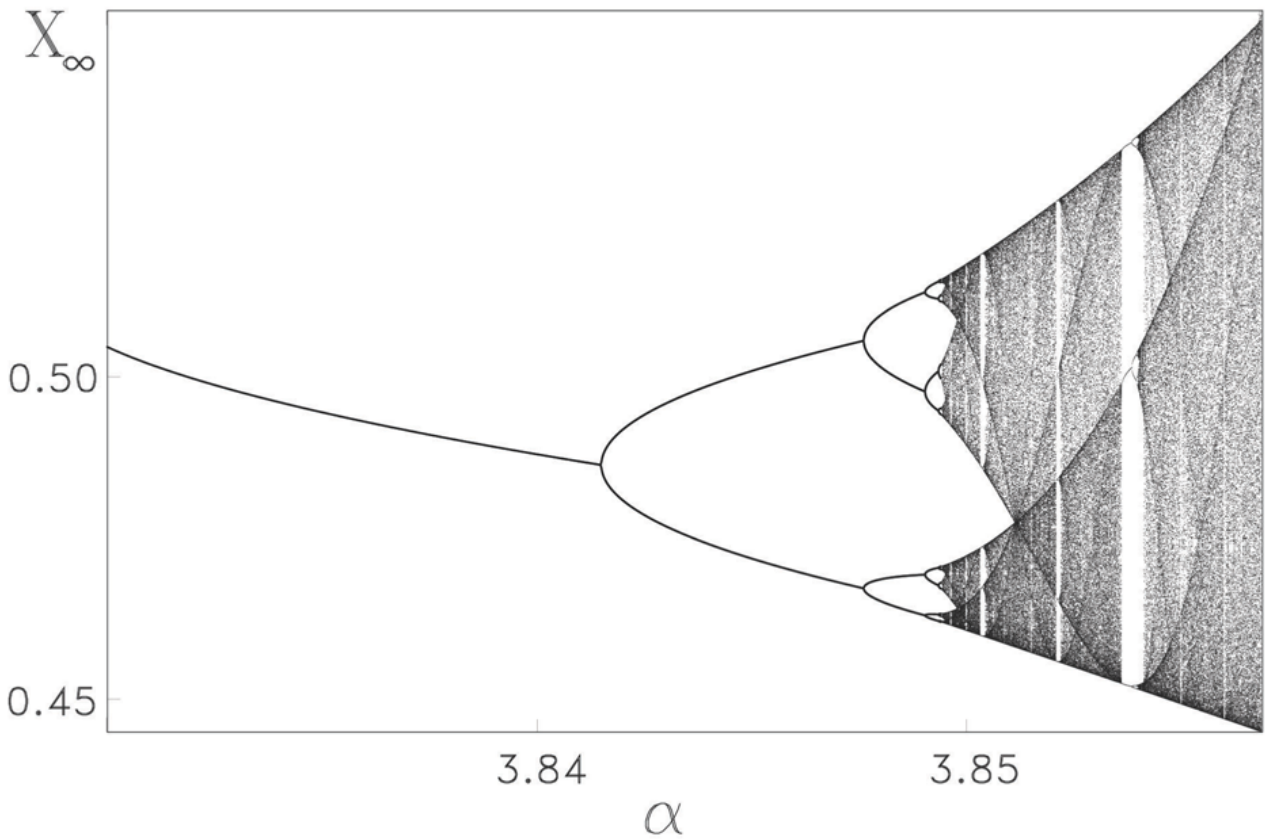
- The tail of the diagram yields: (!)



- The dynamics contain "*white bands*" of periodicity for any number greater than 2. (!)

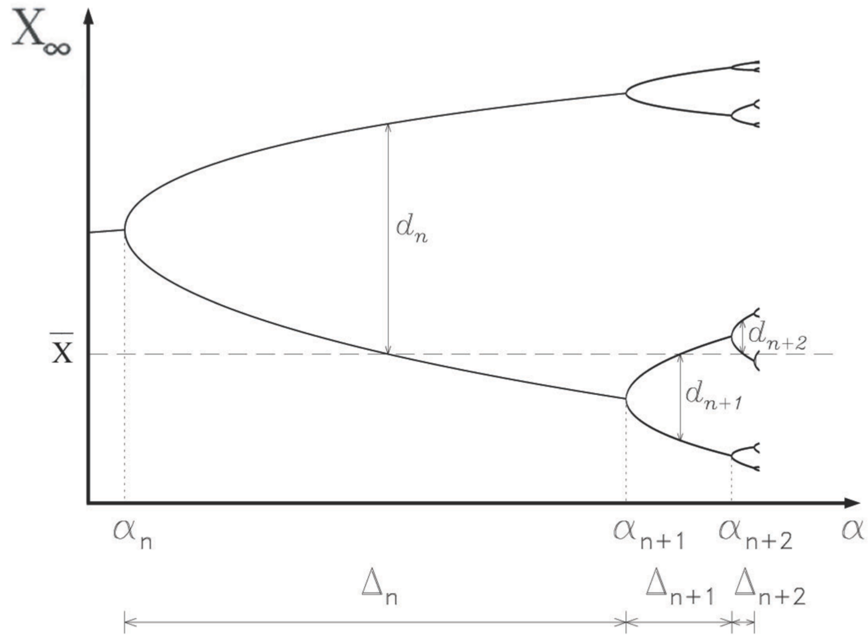
- Amazingly, periodic behavior for whatever number of generations is found in the tree. (!)

- Expanding the middle period 3 "*bud*" gives: (!)



- This is, topologically, just a small copy of the whole tree from its first bent branch. (!)
- The dynamics are unbelievably rich and contain indeed ample *self-similarity*. (!)

- Mitchell J. Feigenbaum showed in 1978 that such behavior is **universal**, as bifurcation openings and frequencies happen according to two *universal* constants, $\mathcal{F}_1$ and $\mathcal{F}_2$:

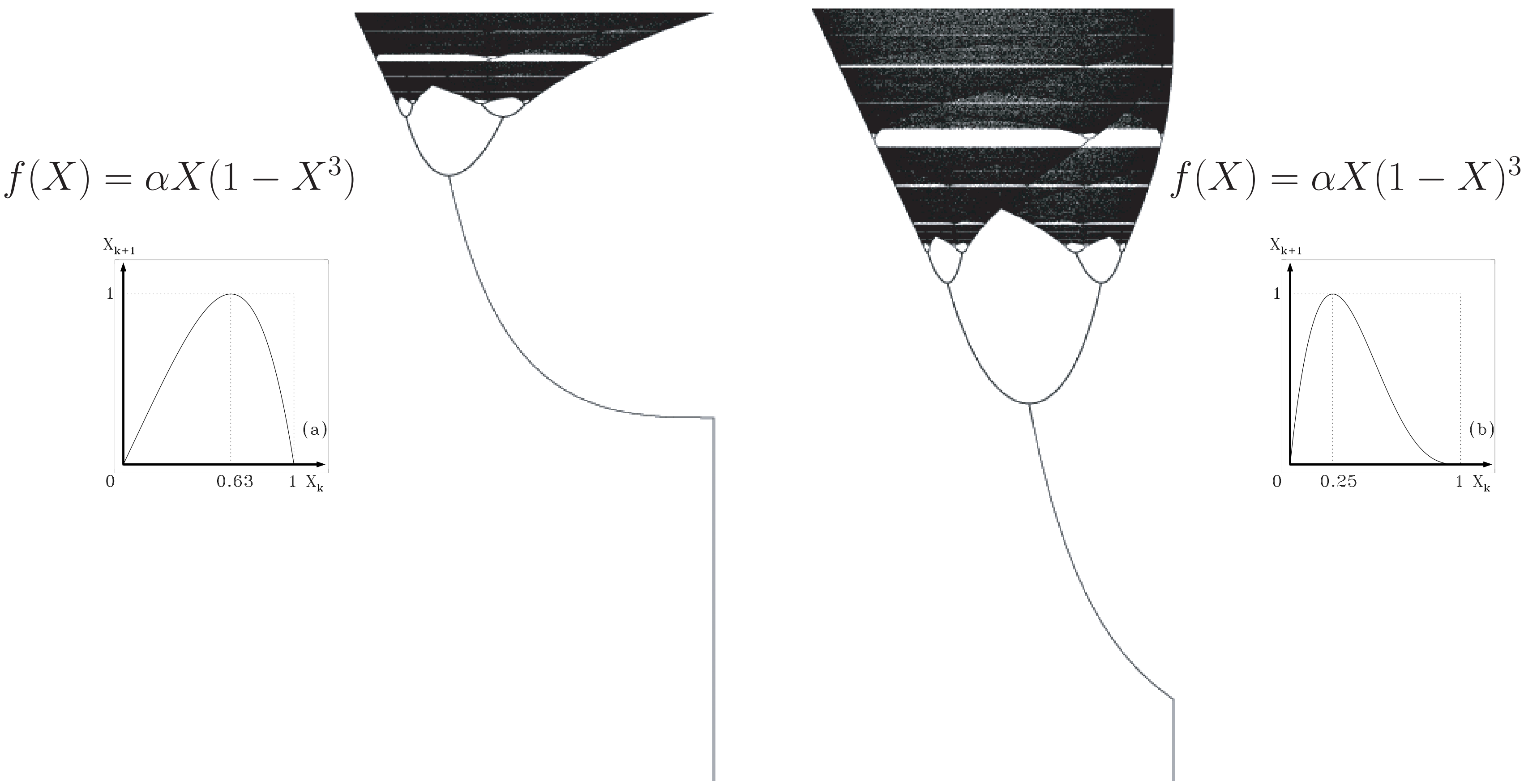


$$\frac{d_n}{d_{n+1}} \longrightarrow \mathcal{F}_1 = -2.5029\cdots, \qquad \frac{\Delta_n}{\Delta_{n+1}} \longrightarrow \mathcal{F}_2 = 4.6692\cdots$$

- The ideas turn out to be relevant in *physics, chemistry, biology, economics* and other fields.

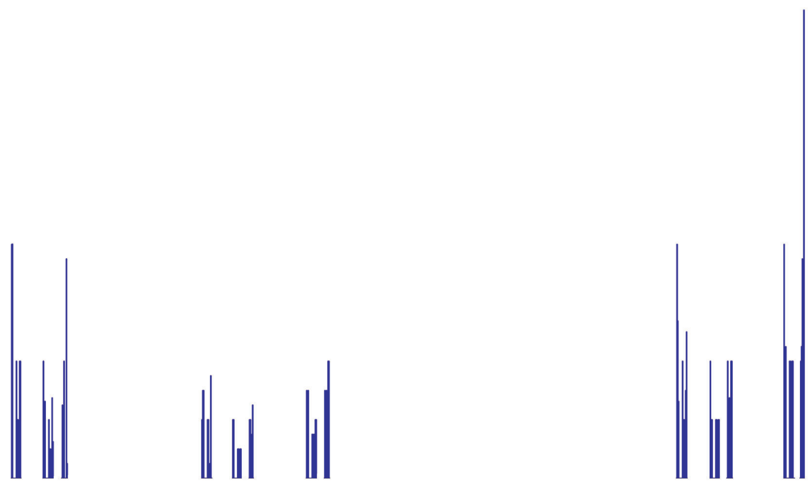
- Interestingly and as found by Albert Libchaber and Jen Maurer for liquid helium in 1978, the pathway to turbulence in the dynamics of **convection** is nicely reproduced by the ideas, when α denotes the amount of *heat* added to a fluid. (!)

- It has beeen established that there is *"order in the way to disorder"* for the *"period-doubling route to chaos:"* for any **non-linear** smooth unimodal map gives a *"root,"* a *"branch,"* bifurcation *"branches,"* and then, in an orderly and intertwined fashion, periodic *"branches"* and the *"foliage of chaos,"* all according to $\mathcal{F}_1$ and $\mathcal{F}_2$: (!)



- The diagrams contain great many places where the dynamics grow **multi-fractal** histograms over *Cantor dust*, similar to the ones described earlier when dealing with cascades.

- For the logistic map, the first one happens at $\alpha_\infty = 3.5699\cdots$, that is, at the so-called *Feigenbaum attractor* made of uncountable dust:

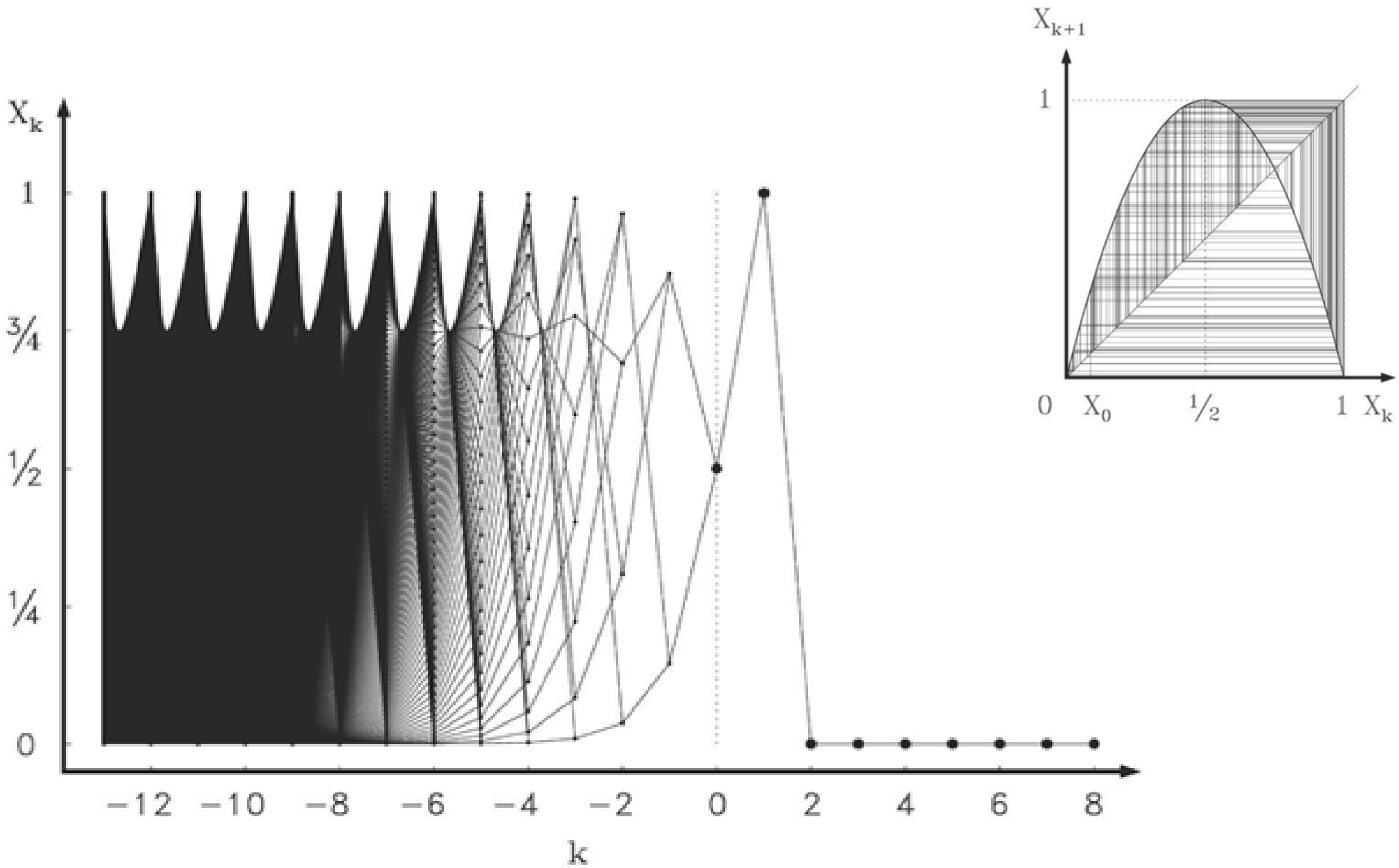


- Similar spiky histograms, with infinitely many *thorns* all going to infinity and satisfying the equation $\infty \cdot 0 = 1$, happen at all *accumulation* points of sprouting buds, at the end of all the periodic windows in the tree. (!)

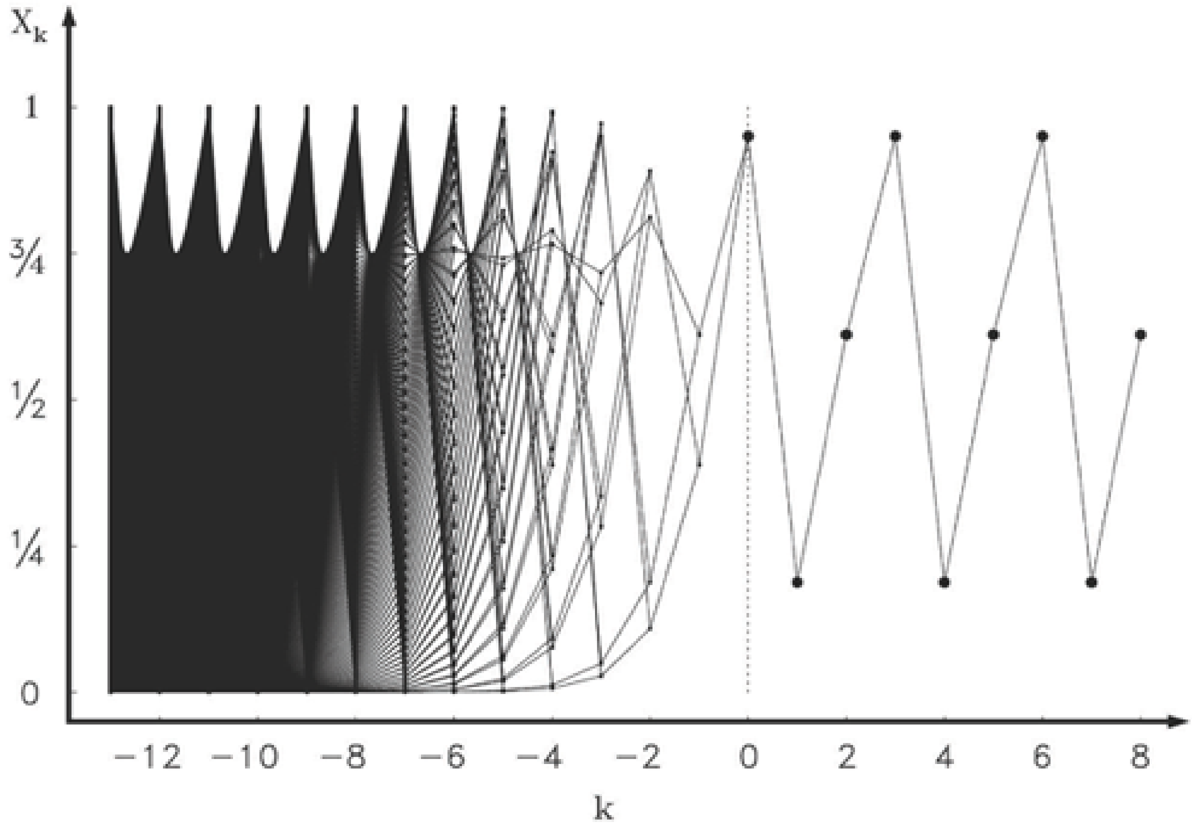
- To further appreciate what chaos is, it is convenient to consider what happens when it is maximum, that is, when $\alpha = 4$ and the parabola's range reaches the interval $[0, 1]$.
- Chaos gives a non-repeating *orbit* that covers almost all the interval $[0, 1]$, but such can not be the whole set for it ought to exclude the *unstable* branches of the tree, as shown on the right below.

- For instance, $X_\infty = 0$, which repeats forever once reached, can not be on the strange attractor and neither can the shown extensions of all periodic points, for such also repeat.

- Not included in the *strange attractor* are also the *pre-images of zero*, i.e., those values that end up at the *origin* and that are found looking backwards on the parabola. For instance, $X_0 = 1$ is not there for $X_1 = 0$, nor $X_0 = 1/2$ for $X_1 = 1$ and $X_2 = 0$, nor the two values that end up in $1/2$, nor those associated with them, folowing an infinite binary tree:



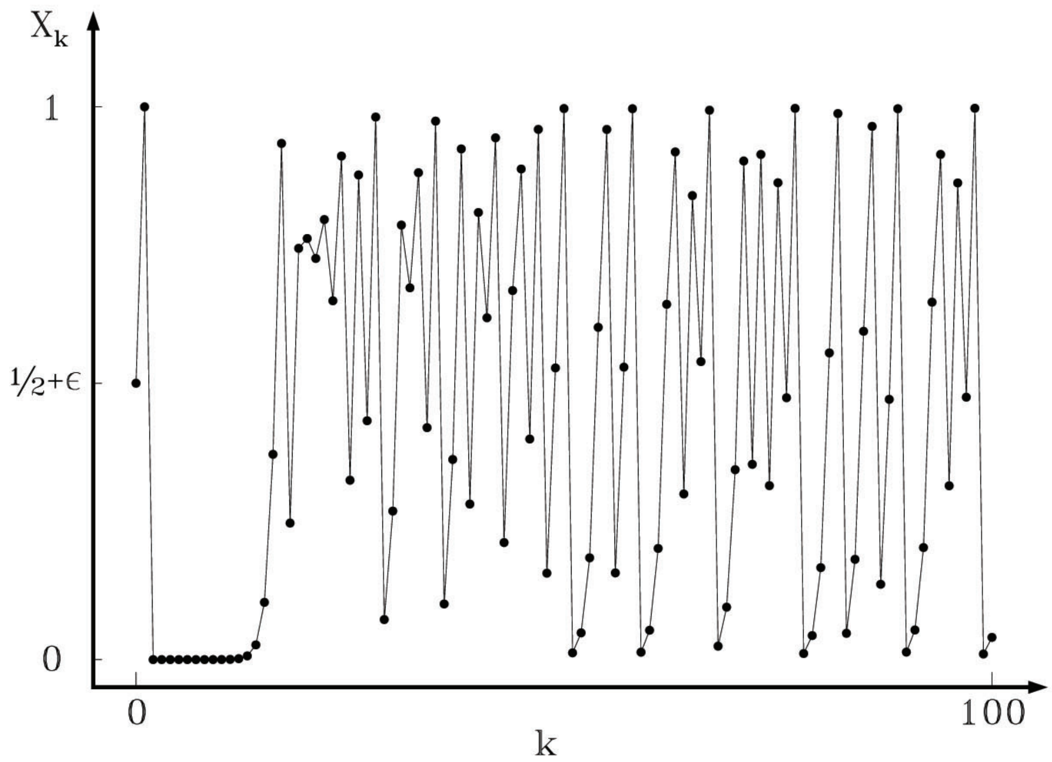
- Also not included are the pre-images associated with the main branch, $X_0 = 3/4$, and also those related to **all** periodic branches that end up *oscillating forever*, as the binary tree below associated with one of the period 3 states:



- Notice how the past looks "*identical*" a few generations before reaching the attractor, and how arbitrarily close initial values then lead to orbits that end up in rather distinct *destinations*.

- If one excludes all such *countable* periodic-related values (trees) (for all periods) from the interval [0, 1], one finally visualizes the *strange attractor*. Such a set is made of a *dense uncountable dust* found everywhere within [0, 1]. (!)

- Chaotic orbits wander forever and always miss the *middle* point. Also, they are prone to *sensitivity to initial conditions*: any small error grows, as shown for $X_0 = 1/2+0.00001$:



- At this time we may pause and wonder what all of these symbols and ideas may mean.
- As the universal concepts in chaos theory remind us of those in turbulence, I believe one may also use them in order to further illustrate our inherent choices.
- We may see that by selecting the "amount of heat" by which we live, we may choose *order* or *disorder*, the *simple* or the *complex*, *serenity* and *peace* or the *turbulent* and *chaotic*.
- As the unforgiving nature of **non-linearities** creeps in in the shoot of the tree, the *Feigenbaum* diagram (**fig tree in German**) reminds us that we may also choose between *decreasing* or *increasing*, and between *attenuating* ($\alpha \leq 1$) or *magnifying* ($\alpha > 1$) the responses to the troubles we face.
- These notions, consistent with what was inferred from turbulence, point to the **root** of the tree, the **Origin**, as the best destination; to "*God's way,*" *below* $X = Y$ rather than "*our way,*" crossing the "just" threshold; and to a condition of "*abandonment,*" **the zero**, and "*obedience*" rather than "*selfishness*" and "*rebelliousness,*" as defined in the Bible.
- For as explained therein, people may receive the blessed *fruitfulness* of a "good choice," that leads to a *home* in "*Heaven,*" or may sadly end up "*cursed*" and sent into a rather painful journey, *wandering* in dust and ever wrestling in the high heat of "*hell.*"

- To conclude, these are some of the options that this lecture reminds us of:

Order	*Disorder*
Simple	*Complex*
Serenity	*Turbulence*
Peace	*Chaos*
Decreasing	*Increasing*
God's way	*Our way*
Below $X = Y$	*Above* $X = Y$
Abandonment	*Selfishness*
Obedience	*Rebelliousness*
Fruitfulness	*Curse*
Home	*Wandering*
Heaven	*Hell*

- The following poem-song summarizes this lesson and draws attention to the next one.

FEIGENBAUM'S PARABEL

In the confines of science
majestically stands a tree,
with all numerals in dance
in emergent chaos to see.

In the instance of a trance
a good day I drew a link,
and here it is, at a glance,
the wisdom that I received.

Foliage of disorder
trapped in empty dust,
jumps astir forever
enduring subtle thrust.

Crossing of the outset
leaving faithful root,
looming tender offset
failing to yield fruit.

Cascade of bifurcations,
increasing heat within,
inescapable succession
of branches bent by wind.

Sprouting of dynamics
attracted to the strange,
oh infinity reminding
at the origin: the flame.

In the midst of chaos
there is a small gate
leading to fine rest.

In the midst of chaos
there are loyal paths
inviting to a dance.

On top of the fig tree
there is a key point
that runs to the core.

On top of the fig tree
there is a clear light
that averts a fright.

In the midst of chaos
there is leaping game
discerning the way.

In the midst of chaos
there is a fine well
watering the brain.

On top of the fig tree
there is a clean frame
that cancels the blame.

On top of the fig tree
there is mighty help
that shelters from hell.

In the midst of chaos,
look it is there,
in the midst of chaos,
logistics in truth,
in the midst of chaos,
a clear faithful route,
in the midst of chaos,
leading to the root.

On top of the fig tree,
this is no delusion,
on top of the fig tree,
a sought needle's eye,
on top of the fig tree,
the symbol of wheat,
on top of the fig tree,
surrounded by weeds.

Could it be, oh my friends,
that science provides a rhyme?,
for a rotten tree foretells
the very advent of time.

Could it be, oh how plain,
that nature extends a call?,
for old parable proclaims
the crux in growing small.

References:

1. H. Bai-Lin (Ed.), *Chaos*, World Scientific, Singapore, 1984.
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4. F. C. Moon, *Chaotic Vibrations*, John Wiley & Sons, New York, 1987.
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