AN INTRODUCTION TO FRACTALS AND COMPLEXITY

Carlos E. Puente

Department of Land, Air and Water Resources

University of California, Davis

http://puente.lawr.ucdavis.edu

Outline

- *Recalls the different kinds of numbers: from naturals to integers, to rationals, to reals.*
- *Reviews the concept of dimension for points, lines, planes and volumes.*
- *Shows examples of fractals, including Cantor dust, Koch curve and Sierpinski triangle.*
- *Contrasts order with chaos via the logistic map.*
- *Introduces natural power-laws and self-organized criticality.*

- Before introducing fractals and other concepts associated with complexity, it is convenient to talk about numbers.

- The first set we learn when we are kids are the **natural** numbers,

$$1, 2, 3, \cdots$$

- This set is *infinite*, and we grasp what "dot, dot dot" means.

- Then, there is the set of **integers**, that include the naturals, zero, and the negatives,

$$\cdots - 2, -1, 0, 1, 2, \cdots$$

- This set is also infinite, but not larger than the naturals, for one can put the integers on a list, i.e., 0 is the *first*, 1 is the *second*, -1 is the *third*, 2 is the *fourth*, -2 is the *fifth*, and so on, "dancing" from left to right and back.

- **Infinity** is indeed an odd concept, for we just showed that

$$2 \cdot \infty + 1 = \infty \quad (!)$$

- The next set of numbers we learn are the **rationals**, the fractions, i.e., the *ratios of integers* denoted by p/q.

- Some examples of these numbers are,

$$\begin{aligned} 1/2 &= 0.5000\cdots \\ 2/3 &= 0.666\cdots \\ 1/11 &= 0.090909\cdots \end{aligned}$$

- As shown, fractions contain a repeatable pattern of digits in their expansion, i.e., the 0's, the 6's or the 09's.

- Sometimes such *"steady state"* is reached immediately, as in $2/3$ and $1/11$, or appears after a finite *"transient,"* e.g., $1/2$ yields a 5 before it settles into infinitely many 0's.

- At the end, the digits of a rational number p/q are fully *predictable*, for its transient and steady-state fully determine the rest of the number's expansion.

- Although expansions are infinite for these numbers, we may easily "rationalize" what "dot, dot, dot" means for them.

- All fractions put together make up another infinite set, but, surprisingly, there are the same number of rationals than naturals, for one can list them sweeping a carpet diagonally:

$$
\begin{array}{cccccccc}
1 & \rightarrow & 2 & & 3 & \rightarrow & 4 & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
1/2 & & 2/2 & & 3/2 & & 4/2 & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & \\
1/3 & & 2/3 & & 3/3 & & 4/3 & \cdots \\
 & \swarrow & & & & & & \\
1/4 & & 2/4 & & 3/4 & & 4/4 & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

- This further confirms that infinity has its own rules, for

$$\infty \cdot \infty = \infty \quad (!)$$

- But not all numbers are fractions, for there are great many others whose decimal expansions do not exhibit finite repetitions.

- These numbers are called **irrationals** and there are so many of them that they *can not even be listed*, i.e., they are associated with a "larger" infinity. (!)

- Prominently among them there are

$$\begin{aligned} \sqrt{2} &= 1.41421356\cdots \\ \pi &= 3.14159265\cdots \\ e &= 2.71828183\cdots \end{aligned}$$

  that are associated with **squares**, **circles** and **spirals**.

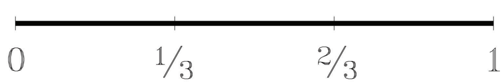
- As the expansions above do not repeat, one *can not predict* the next digit. Hence, "dot, dot, dot" for these numbers describe an internal *"mystery."*

- In fact, the digits for these and all irrationals are so "disorganized" that it appears to us that they are *"guided by chance."*

- As infinite expansions represent an unbridgeable limitation, irrational numbers may only be fully understood if they possess a particularly defining property, like the three famous numbers above.

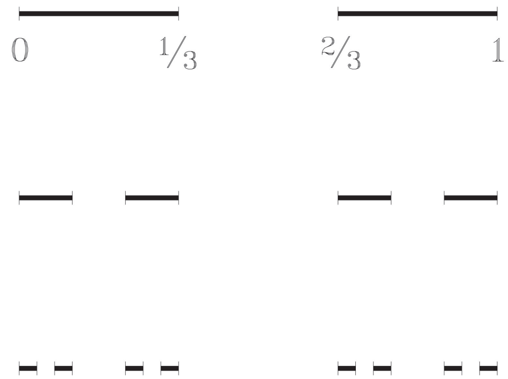
- The irrationals and the rationals together make the **real** numbers. These are the collection of "points" that are represented on a one-dimensional line.

- Now, with numbers fully reviewed, we may turn our attention to the concept of **dimension**.
- We know that a point has no dimension, that a line segment is one-dimensional, that a plane is two-dimensional, and that we live in three-dimensional space.
- It happens that there is an easy way to verify such results, by counting the number of *"boxes"* required to cover a given set.
- Consider, for instance, intervals of a size δ and ask how many of them, $N(\delta)$, are required to cover a line segment having size 1.
- If δ is equal to 1, then clearly one interval suffices, i.e., $N(1) = 1$. If $\delta = 1/3$, then one requires 3 such intervals, i.e., $N(1/3) = 3$:



- Clearly, if $\delta = 1/n$, then $N(\delta) = n$, and this leads to a simple relationship between δ and $N(\delta)$, namely, $N(\delta) = \delta^{-1}$. It turns out that the inverse of the obtained exponent yields the *dimension* of the line segment, i.e., $D = 1$. (!)

- That the ideas works for a point may be easily verified, for irrespective of δ one always requires 1 such interval in order to cover a point, i.e., $N(\delta) = \delta^0 = 1$, and hence $D = 0$.
- For a plane or a volume the arguments are similar, but instead of using intervals to cover such sets it becomes appropriate to employ squares of cubes of a given *side* δ.
- Clearly, for a plane one gets $N(\delta) = \delta^{-2}$, for a reduction of δ by a factor of two yields a four-fold, 2^2, increase in the number of required squares, as may be easily verified glancing at floor tiles.
- Likewise it happens for a cube, $D = 3$, for a change in δ by a factor of two results in an eight-fold, 2^3, increase in $N(\delta)$.
- The aforementioned sets are prototypical *Euclidean* objects.
- **Fractals** are "fragmented" geometric sets whose *fractal dimensions*, as defined counting pieces as before, are typically non-integers that exceed their topological dimensions.
- As an example, consider the so-called *Cantor dust* defined as the remains of recursively "taking open third subintervals" from a given interval of size 1, as illustrated in the following sketch.



- As may be hinted, the Cantor set is made of infinitely many points. It clearly contains the countable corners on the little intervals, but it turns out to be uncountable in size, as it defined by all real numbers within $[0, 1]$ whose ternary expansion, in terms of 0's, 1's and 2's, does not contain a 1. (!)

- Topologically speaking, this set is just *"sparse dust,"* but, as there are great many points, its dimension turns out to be greater than zero.

- Calculation of such a quantity may be done in parallel to what was explained earlier for a line segment, as follows.

- If one chooses an interval of size 1, then clearly it may be used to cover the whole Cantor set, i.e., $N(1) = 1$. As δ is dropped to $1/3$, then $N(\delta) = 2$; as $\delta = 1/9$, $N(\delta) = 4$, and so on.

- In general, $N(1/3^n) = 2^n$, and this leads, after a little algebra, to $N(\delta) = \delta^{-D}$, where the exponent is given in terms of natural logarithms, $D = ln2/ln3 \approx 0.63$. (!)

- The Cantor set is a *fractal* set because its dimension exceeds its *topological* dimension of zero.

- Notice that other kinds of *dusts* may be easily constructed just by varying the size of the original hole in the interval.

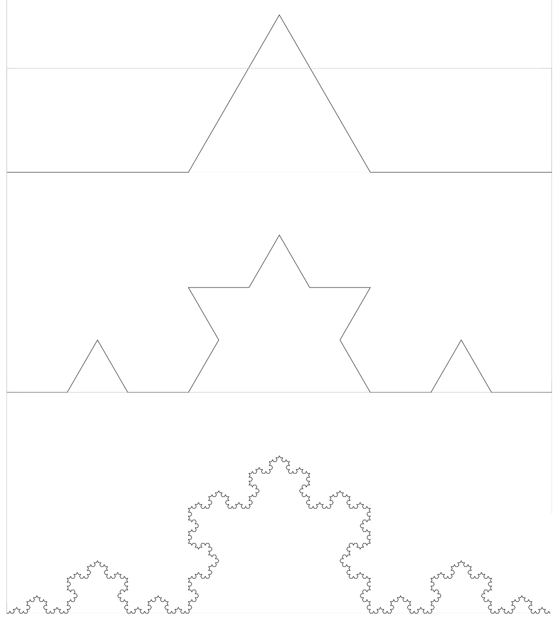
- If instead of removing middle third subintervals one takes out l% equidistant segments, the implied dimension becomes $D = ln2/(ln2 - ln(1 - l))$.

- Such a dimension reflects the amount of *space covered* by the dust, for, depending on the size of l, it could be any number between 0 and 1.

- When $l = 0$ the resulting set is, by construction, the interval from 0 to 1, and clearly such a set has dimension 1. As l increases towards 1, the obtained set is increasingly sparse and its dimension decreases towards 0.

- Fractals may also be defined over two- or three-dimensional space, as follows.

- The *Koch curve*, found replacing every line segment by four smaller segments having a third of the original length and making up through the middle an outer equilateral triangle,

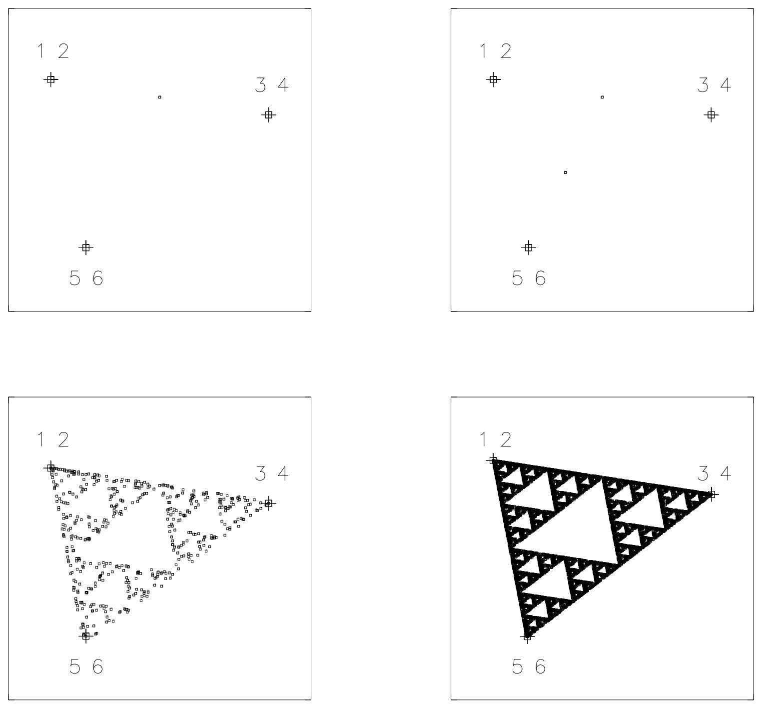


  is a fractal set on two dimensions that has fractal dimension $D = ln4/ln3 \approx 1.26$.

- Other sets, topologically one-dimensional, and having dimensions between 1 and 2 (inclusive) may be constructed just by varying the construction rule.
- The Koch curve, and others with dimensions greater than one, fill-up space in varying degrees due to their *infiniteness*.

- Another celebrated fractal set in the plane and having the same dimension of the Koch curve is the *Sierpinski triangle*, found removing (in the spirit of the construction of the Cantor set) successive middle triangles from a given solid triangle.

- Such a set may be obtained randomly *iterating* three simple rules that move a given location to half the distance from the vertices of the original triangle:



- Although distinct looking sets may have the same dimension, the notion of fractals has provided a suitable framework to address nature's complex geometries, in one and higher dimensions.

- For as stated by Benoit Mandelbrot, who first coined the word fractal, "clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travels in a straight line." (!)

- Even though nature does not provide the precise repetitiveness, i.e., *self-similarity*, as shown in the examples, it is well established by now that fractals are relevant to many scientific fields that include physics, geophysics, economics, and biology.

- Fractals are indeed everywhere, as the fabric of nature often results in fragmentation via the repetition of simple rules, for counting the number of boxes needed to cover many natural sets define *power-laws*, i.e., $N(\delta) \sim \delta^{-D}$. (!)

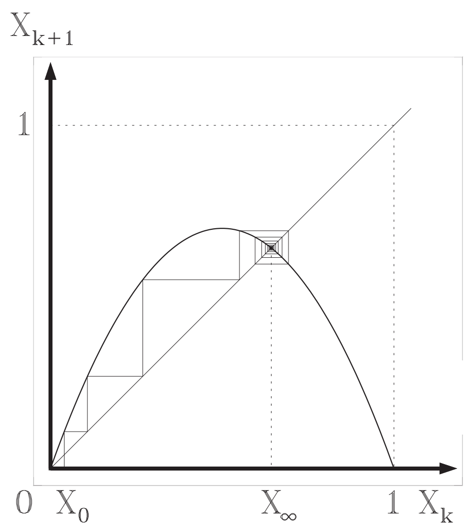
- Fractals sets have also been found associated with the dynamics of *non-linear* systems.

- Such include the complex unpredictable behavior known as **chaos**.

- To illustrate what chaos is, it is pertinent to study the equation $X_{k+1} = \alpha X_k(1 - X_k)$, denoting the evolution of a (normalized) population from a generation to the next.

- This equation, known as the *logistic map* and shaped as a parabola, results in alternative behavior depending on the value of the *parameter* α, a number between 0 and 4.

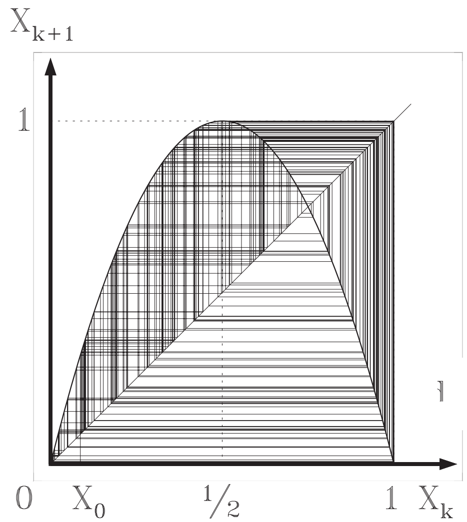
- When $\alpha = 2.8$ the population rests at X_∞, the non-zero intersection of the 45 degree line and the parabola,



irrespective of the initial population value denoted by X_0.

- This case corresponds to an *ordered* condition, nicely expressed by the expansion of a *rational* number with a single *steady-state*. (!)

- When $\alpha = 4$ the population does not rest at all, but wanders in dust for ever without any repetition on a *fractal attractor*,



- This case is essentially *unpredictable* and its structure is captured by the expansion of an *irrational* number. (!)

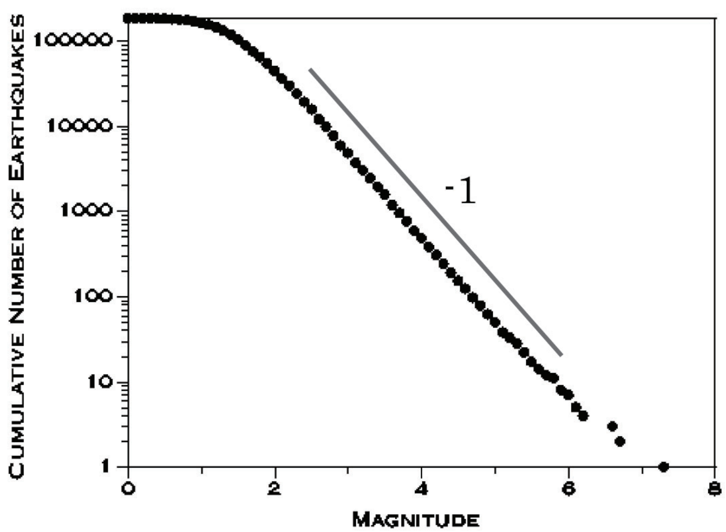
- This condition is known as *chaotic* for a small error in X_0 yields sizable variations, e.g., whereas an initial value of 0.4 results in 0.1 after 7 generations, an initial value of 0.41 yields instead 0.69.

- Chaos was first recognized while studying the dynamics of the weather. Its presence represented a major breakthrough for it established that complexity may have simple roots.

- A common trait of natural complex phenomena is the presence of *power-laws* in the frequency distribution of events,

$$P[X \geq x] \sim x^{-c}$$

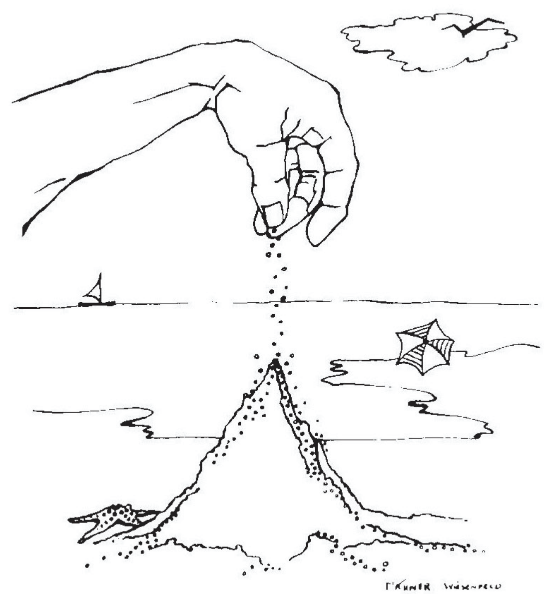
- As in fractal sets, the *simple log-log "lines"* reflect the absence of *characteristic scales* in many natural processes, such as earthquakes, floods, avalanches, fires, etc. (!)



- Such distributions possess *"heavy tails"* relative to the *normal* or *Gaussian* distribution, associated with *independence*.

- A very good model for such behavior in a variety of natural processes is the concept of *self-organized criticality*, as introduced by Per Bak.

- The metaphor is that systems, via the *accumulation of energies*, arrange into a critical state always at the verge of disintegration, as illustrated via a sand pile. Here, *avalanches* of various sizes happen according to a power-law. (!)



- The ideas in this introduction and their relation to **love** and **peace** shall be further elaborated in subsequent lectures.

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